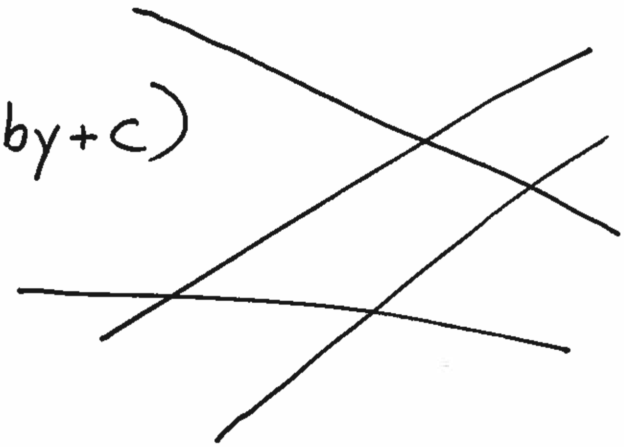
Lecture 34: Projective Space.

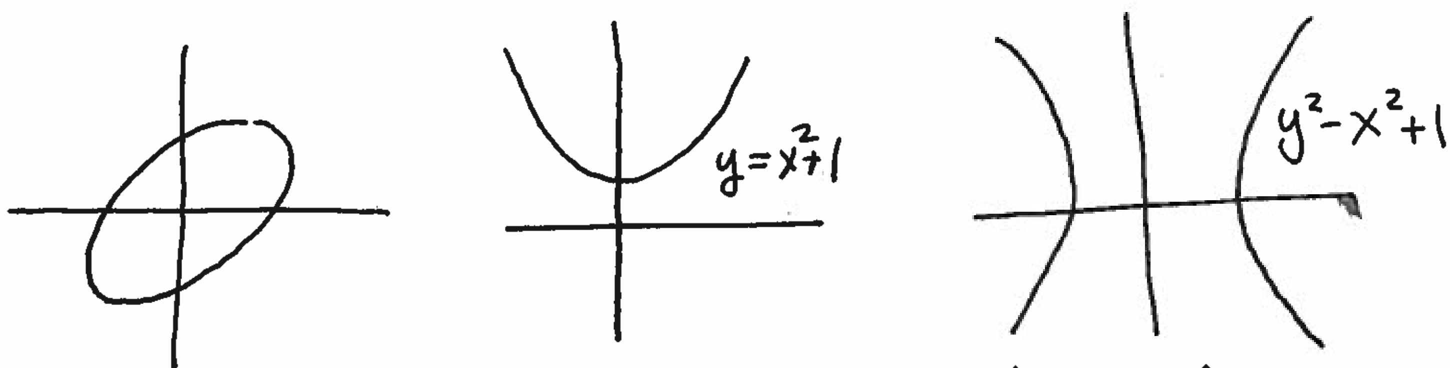
Inconvenient truths:

① Lines in $\mathbb{R}^2$: $V = \mathbb{V}(ax+by+c)$

Two distinct lines intersect in one pt, except when they don't, i.e. are parallel.

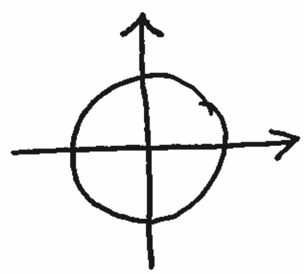


② Plane Conics in $\mathbb{R}^2$: $V = \mathbb{V}(ax^2+bxy+cy^2+dx+ey+f)$



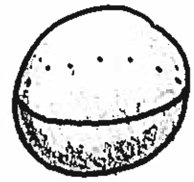
Ellipse Parabola Hyperbola

Wouldn't it nice if they were all the same?

③ $\mathbb{V}(x^2+y^2-1) \subseteq \mathbb{R}^2 \Rightarrow$ 

What is

$$\mathbb{V}(x^2+y^2-1) \subseteq \mathbb{C}^2 \,?$$

It should have $\dim_{\mathbb{C}} = 1$ and $\dim_{\mathbb{R}} = 2$, so a reasonable guess is: $V =$ 

<u>But</u>: V is not compact, since $\forall a \in \mathbb{C}$ we can solve $a^2 + y^2 = 1$ to find a point $(a,b) \in V$; thus V is <u>not</u> bounded in $\mathbb{C}^2 \cong \mathbb{R}^4$.

<u>The Fix</u>: Projective Space

For a field k, ~~define~~ will have $\mathbb{P}^n_k = k^n \cup \left\{ \begin{matrix} \text{stuff at} \\ \infty \end{matrix} \right\}$

$\quad$ ↑ Affine Space

Start with $\mathbb{P}^2_{\mathbb{R}}$, the <u>projective plane</u>.

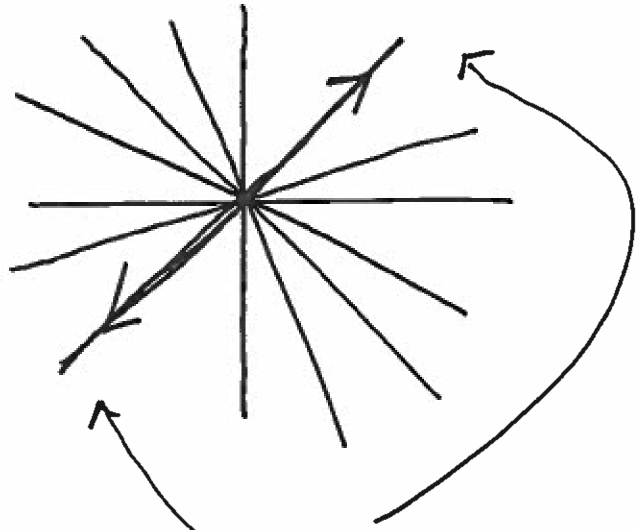
Need to add pts at ∞ so that

ⓐ Any two parallel lines meet at ∞.

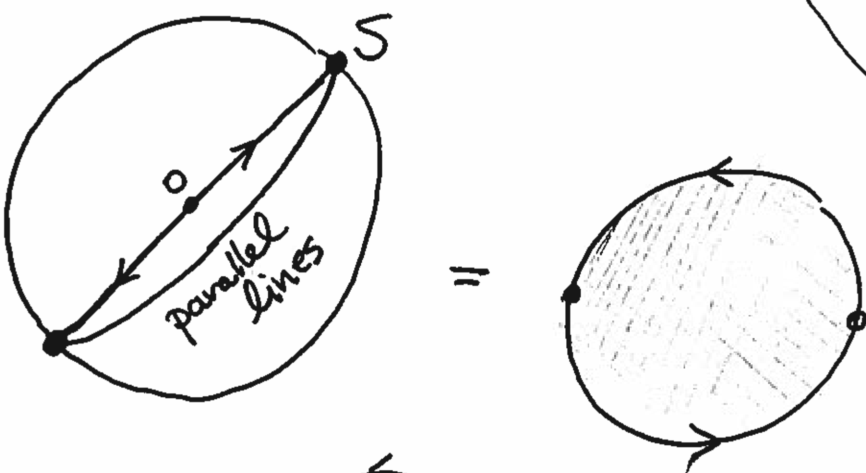
ⓑ Any two nonparallel lines <u>don't</u> meet at ∞.

<u>Idea</u>: $S = \mathbb{P}^2_{\mathbb{R}} \setminus \mathbb{R}^2$ ~~is~~ has one pt for each line through 0.

Note S is a circle:



Hence $\mathbb{P}^2_{\mathbb{R}}$ looks like



Compare:

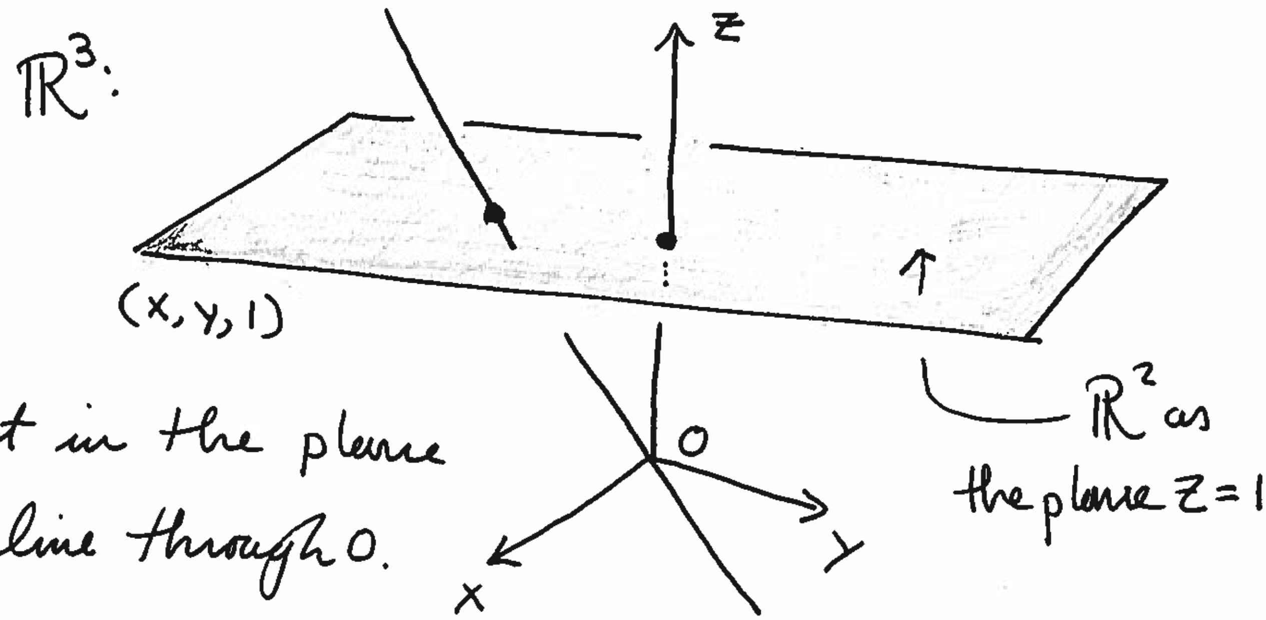
Def: $\mathbb{P}^2_{\mathbb{R}} = \{\text{lines through } 0 \text{ in } \mathbb{R}^3\}$

$$= \{(x,y,z) \in \mathbb{R}^3 \setminus \{0\}\} \Big/ (x,y,z) \sim (\lambda x, \lambda y, \lambda z) \text{ for } \lambda \in \mathbb{R}^\times$$

Points in $\mathbb{P}^2_{\mathbb{R}}$ will be denoted $(x:y:z)$

Observations:

$\mathbb{R}^2 \subseteq \mathbb{P}^2_{\mathbb{R}}$ as $\{(x:y:1) \mid x,y \in \mathbb{R}\}$ Any pt has at most one rep of this form



Each pt in the plane det. a line through 0.

What's left?

$$\mathbb{P}^2_{\mathbb{R}} \setminus \mathbb{R}^2 = \{(x:y:0) \mid x,y \text{ not both } 0\}$$

$$= \{\text{lines through } 0 \text{ in } \mathbb{R}^2\} = \mathbb{P}^1_{\mathbb{R}} \quad \text{"Circle at } \infty.\text{"}$$

For any k, n define

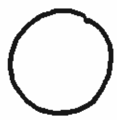
$$\mathbb{P}^n_k = \{\text{lines through } 0 \text{ in } k^{n+1}\}$$

$$= \{a \in k^n \setminus \{0\}\} \big/ a \sim \lambda a \text{ for } \lambda \in k^\times$$

which contains $k^n = \mathbb{A}^n_k$ as $\{(a_1 : a_2 : \cdots : a_n : 1\}$

with $\mathbb{P}^n_k \setminus \mathbb{A}^n_k = \mathbb{P}^{n-1}_k$.

Ex: $\mathbb{P}^1_k = k \cup \{(1:0)\}$ ← the pt at ∞.

$\mathbb{P}^1_{\mathbb{R}} =$  $\quad \mathbb{P}^1_{\mathbb{C}} =$ 

the Riemann Sphere.

$V \subseteq k^n$ will now be called <u>affine varieties</u> in contrast to <u>projective varieties</u> in $\mathbb{P}^n_k$.

<u>Q</u>: How do we even make sense of poly equations on $\mathbb{P}^2_{\mathbb{R}}$??

<u>1^{st} attempt</u>: Take poly in coor $(x:y:z)$, e.g.

$f = xy - z \qquad f(1:1:1) = 0$

$\qquad\qquad\qquad f(2:2:2) = 2$

<u>Def</u>: A polynomial $f \in \mathbb{R}[x,y,z]$ is <u>homogenous</u> if the degree $(x^a y^b z^c \to a+b+c)$ of all terms are the same.

<u>Ex</u>: $xy - z^2$ <u>Non Ex</u>: $xy - z$

Suppose $f \in \mathbb{R}[x,y,z]$ is homogenous. Define

$$\mathbb{V}(f) = \{ (a:b:c) \in \mathbb{P}^2_{\mathbb{R}} \mid f(a,b,c) = 0 \}$$

which makes sense because for $\lambda \neq 0$ we have

$$f(\lambda a, \lambda b, \lambda c) = \lambda^n f(a,b,c)$$

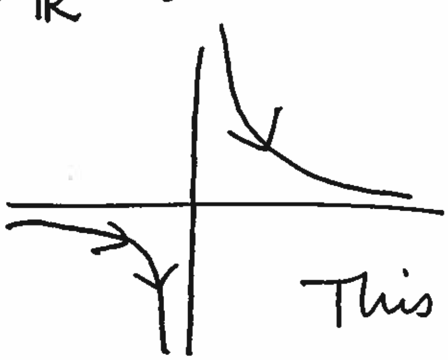
where n is the degree of f.

<u>Ex</u>: $\mathbb{V}(xy - z^2)$. What is $V \cap \mathbb{R}^2$?

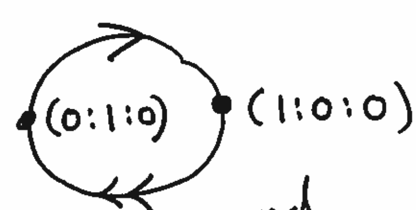
$$= \{ (x:y:1) \mid xy - 1 = 0 \} = \mathbb{V}_{\mathbb{R}^2}(xy-1)$$

What is $V \cap \mathbb{P}^1_{\mathbb{R}}$? $\{ (x:y:0) \mid xy = 0 \} = \left\{ \begin{matrix} (1:0:0) \\ (0:1:0) \end{matrix} \right\}$

<u>Note</u>:



$V =$



This surmounts the 2nd inconvenience.