Lecture 38:

$V \subseteq k^n$ an <u>irreducible</u> affine variety

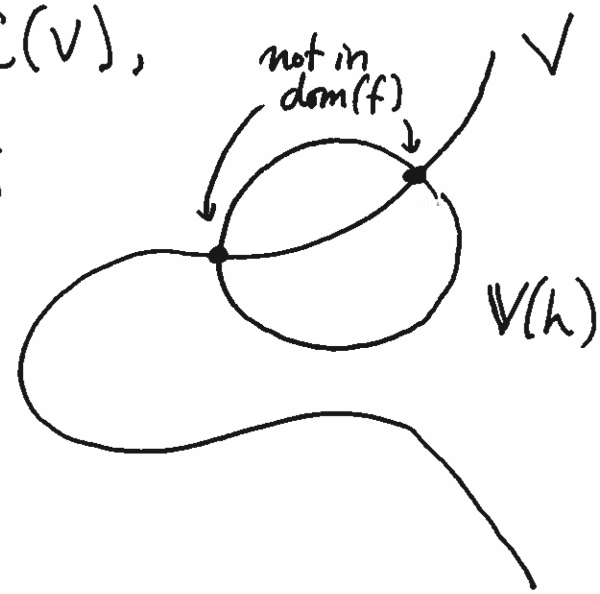
$k(V)$ = function field = field 'o fractions of $k[V]$

For $f \in k(V)$, called a <u>rational function</u>, set

$$\mathrm{dom}(f) = \left\{ p \in V \;\middle|\; \begin{array}{l} \text{Can rep } f \text{ as } \frac{g}{h} \text{ with} \\ h(p) \neq 0 \end{array} \right\}$$

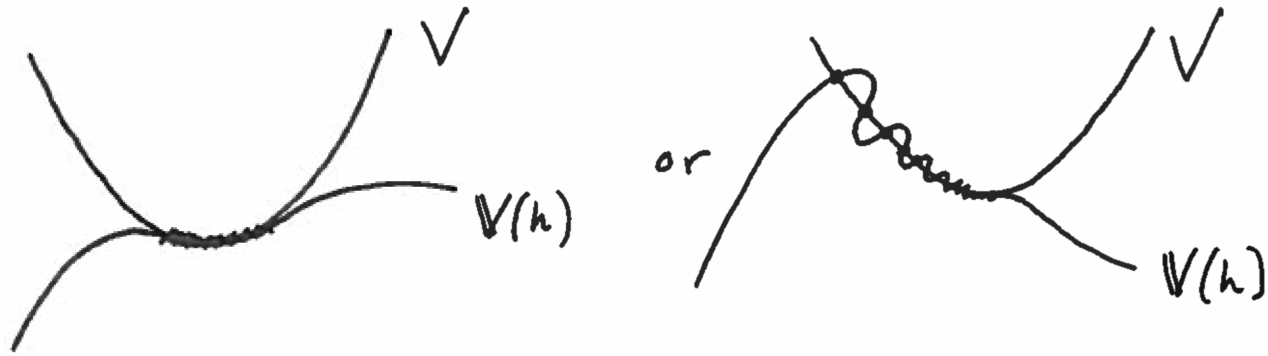
<u>Prop</u>: Suppose $V = \mathbb{V}(r(x)) \subseteq \mathbb{C}^2$ is a smooth plane curve. Then for any $f \in \mathbb{C}(V)$,

$$\mathrm{dom}(f) = V \setminus \{\text{finite set}\}$$



<u>Idea</u>: If $f = \dfrac{g}{h}$ need to show $V' = \mathbb{V}(r, h) = $ finite set

<u>Moral</u>: Polynomials are determined locally, not too complicated. So can't have:



<u>Similar Fact</u>: $f, g \in \mathbb{C}[z]$. If $\exists\, \varepsilon > 0$, $z_0 \in \mathbb{C}$ so that $f(z) = g(z)$ for all $z \in B_\varepsilon(z_0)$, then $f = g$ in $\mathbb{C}[z]$.

<u>Pf.</u> Can assume $z_0 = 0$. As $f = g$ on $B_\varepsilon(0)$, have $f^{(n)}(0) = g^{(n)}(0)$ for all n. So f and g have the same coeff on z^n, namely $\frac{1}{n!} f^{(n)}(0)$, so $f = g$ in $\mathbb{C}[z]$

<u>Comforting fact</u>: If V is a smooth irreducible plane curve in $\mathbb{C}^2$, then any $f \in \mathbb{C}(V)$ comes from an honest continuous function

$$\bar{f} : V \longrightarrow \mathbb{P}^1_{\mathbb{C}} \quad \text{where } \bar{f}(p) = \infty \iff p \notin \mathrm{dom}(f).$$

Ex: $V = \mathbb{C}$, $\mathbb{C}(V) = \mathbb{C}(t)$, and $f \in \mathbb{C}(V)$ has the form:

$$f = c\,\frac{(t-a_1)\cdots(t-a_k)}{(t-b_1)\cdots(t-b_\ell)}$$

with $\operatorname{dom}(f) = \mathbb{C} \setminus \{b_1, \dots, b_\ell\}$.

[Goal: solve the inverse Galois prob for $\mathbb{C}(t)$...]

Let V be a ~~smooth~~ irred plane curve in $\mathbb{C}^2$, and $h \in \mathbb{C}[V]$ be a poly. fn. From

$$h : V \longrightarrow \mathbb{C}$$

we get an induced ring homomorphism:

$$\mathbb{C}[V] \xleftarrow{\;h^*\;} \mathbb{C}[t] = \mathbb{C}[\mathbb{C}]$$

via

$$h^*(f) = f \circ h$$

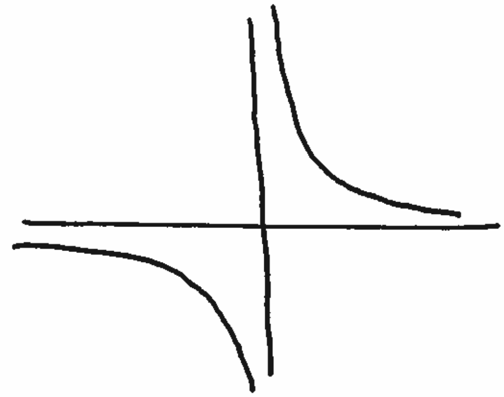
That is:

$$h^*(f)(x,y) = f(h(x,y))$$

<u>Ex</u>: $V = \mathbb{V}(xy-1)$

$h = x+y \in \mathbb{C}[V]$

$f(t) = t^2+t+1 \in \mathbb{C}[t]$



$$h^*(f) = (x+y)^2 + (x+y)+1 = x^2 + \overbrace{2xy}^{=2 \text{ on } V} + y^2 + x + y + 1$$
$$= x^2+y^2+x+y+3$$

In general, what is $\ker(h^*)$? Suppose $f \in \mathbb{C}[t]$ is nonzero. If $h^*(f) = 0$, then $f(h(x,y)) = 0$ in $\mathbb{C}[V]$ $\Rightarrow$ Every point of $h(V)$ is a root of f $\Rightarrow$ $h(V)$ is finite $\Rightarrow$ (as V is irreducible) $h(V) =$ one pt $\Rightarrow$ h is ~~the~~ a constant map, coming from a poly with only a const term.

<u>Prop</u>: If h is non-constant, then $\ker(h^*) = 0$.

Define a field homom. $h^*: \mathbb{C}(t) \longrightarrow \mathbb{C}(V)$

by
$$h^*\left(\frac{p(t)}{q(t)}\right) = \frac{h^*(p(t))}{h^*(q(t))} = \frac{p(h(x,y))}{q(h(x,y))}$$

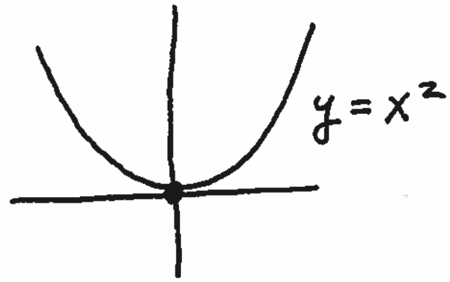
By the prop, as long h is non-constant this is well-defined as $q(t) \neq 0 \Rightarrow h^*(q(t)) \neq 0$.

As $h^*(1) = 1 \neq 0$, the field homom h^* is nontrivial and hence 1-1; thus we have

$$\mathbb{C}(t) \underset{h^*}{\hookrightarrow} \mathbb{C}(V)$$

i.e. we have a field extension $\mathbb{C}(V)/\mathbb{C}(t)$.

Ex: $V = \mathbb{V}(y - x^2)$



Consider $h(x,y) = y \in \mathbb{C}[V]$ as a fn $V \xrightarrow[h]{} \mathbb{C} = \{y\text{-axis}\}$.

The ring homom $\mathbb{C}[t] \xrightarrow{h^*} \mathbb{C}[V]$ is $h^*(f(t)) = f(h(x,y))$
$$t \longmapsto y \qquad\qquad = f(y).$$

As h is non-constant, get an injective field hom.

$$\mathbb{C}(t) \hookrightarrow \mathbb{C}(V)$$
$$t \longmapsto y$$

Can be read either way!

Identify $\mathbb{C}(t)$ with its image $\mathbb{C}(y) \subseteq \mathbb{C}(V)$ and call it F. Set $K = \mathbb{C}(V)$. What is K/F?

1. K/F is simple, since $K = F(x)$
2. K/F is algebraic, as x is a root of $z^2 - y \in F[z]$
3. $z^2 - y$ is irreducible (Eisenstein with $R = \mathbb{C}[y]$, $I = (y)$)

So $K = F[z]/(z^2 - y) = F(\sqrt{y})$

<u>Fun Fact</u>: As abstract fields, $K \cong F$! [project onto x-axis instead]