## Lecture the last

**Thm**: $G$ finite group. Then $\exists$ a Galois extension $K$ of $\mathbb{C}(t)$ with $\mathrm{Gal}(K/\mathbb{C}(t)) = G$.

**Plan**: ① Find an irreducible smooth curve $V$ in $\mathbb{P}^n_{\mathbb{C}}$ on which $G$ acts by symmetries, and where $V/G = \mathbb{P}^1_{\mathbb{C}}$.

② Then acts on $K = \mathbb{C}(V)$ by $\sigma \in G \longmapsto \sigma^*$ where we view $\sigma : V \to V$ and $\sigma^*(f) = f \circ \sigma^{-1}$

③ Then $K_G = \mathbb{C}(V/G) = \mathbb{C}(t)$. As always, $K/K_G$ is Galois with group $G$.

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Last time, given $G$ we constructed an action on a surface $Y$ via

Thicken, add discs

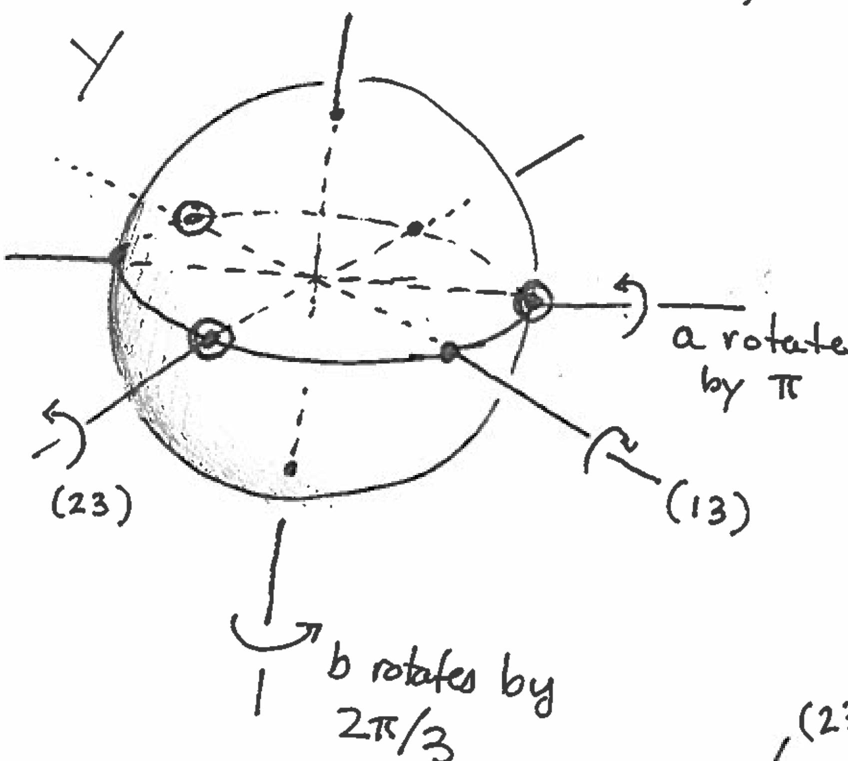
$$G, S \text{ gen. set} \rightsquigarrow \Gamma(G, S) \rightsquigarrow Y$$

Cayley Graph

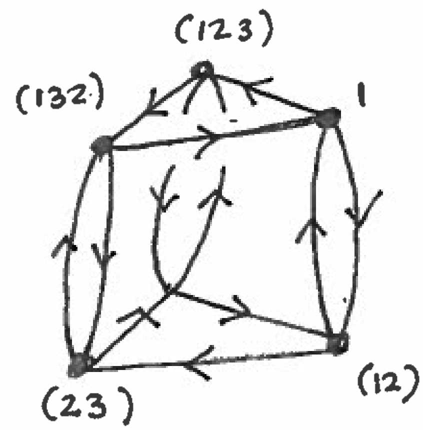
and where $Y/G = X =$ .

[Did this for $S_3$, but construction is actually completely general. $Y$ typically isn't a sphere, though.]

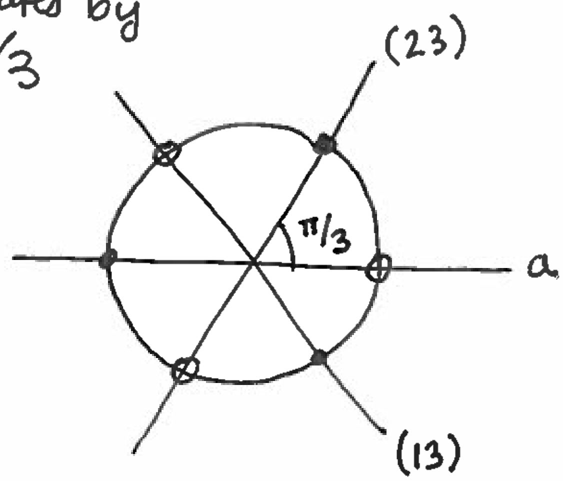
Ex: $G = S_3 = \langle a = (12),\ b = (123) \rangle$




Note: Picture rotated compared to last class



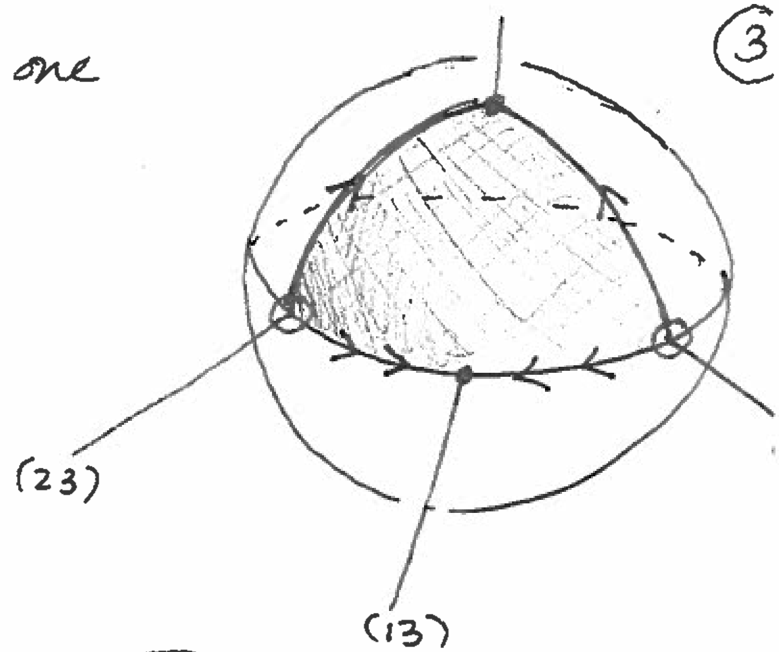

View from above:

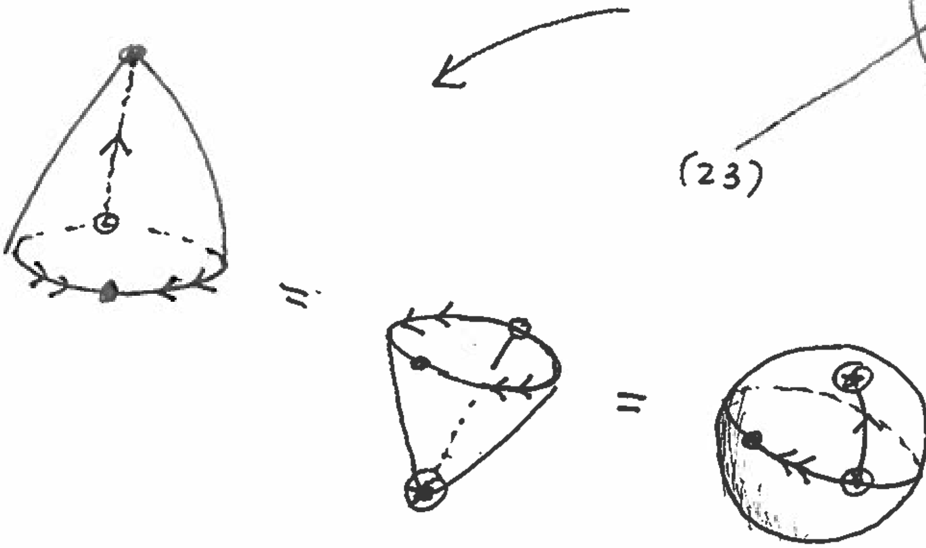



From this point of view, how do we work out $X = Y/G$?

[We know the answer from last time, but this is helpful to understand what's going on.]

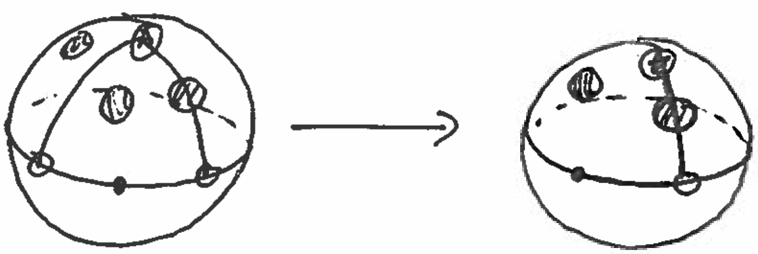
Every point is equivalent to one in the shaded region:



$= X$

Let $B$ be the 8 points on $Y =$ . Away from $B$, the map $\pi : Y \longrightarrow X$ is locally 1-1.



Near the poles, the map looks like $z \mapsto z^3$

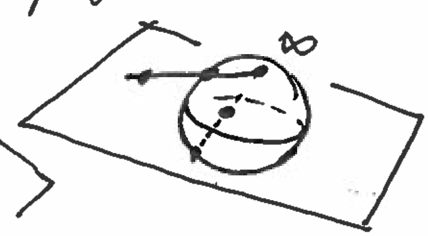
Near the other points of $B$, looks like $z \mapsto z^2$.

<u>So</u>: We've found a surface $Y$ with symmetries $G$ with $X = Y/G =$ . So we have a <u>continuous</u> map $\pi : Y \longrightarrow X$ which looks locally like a polynomial map. [In topology, this is called a branched cover.]

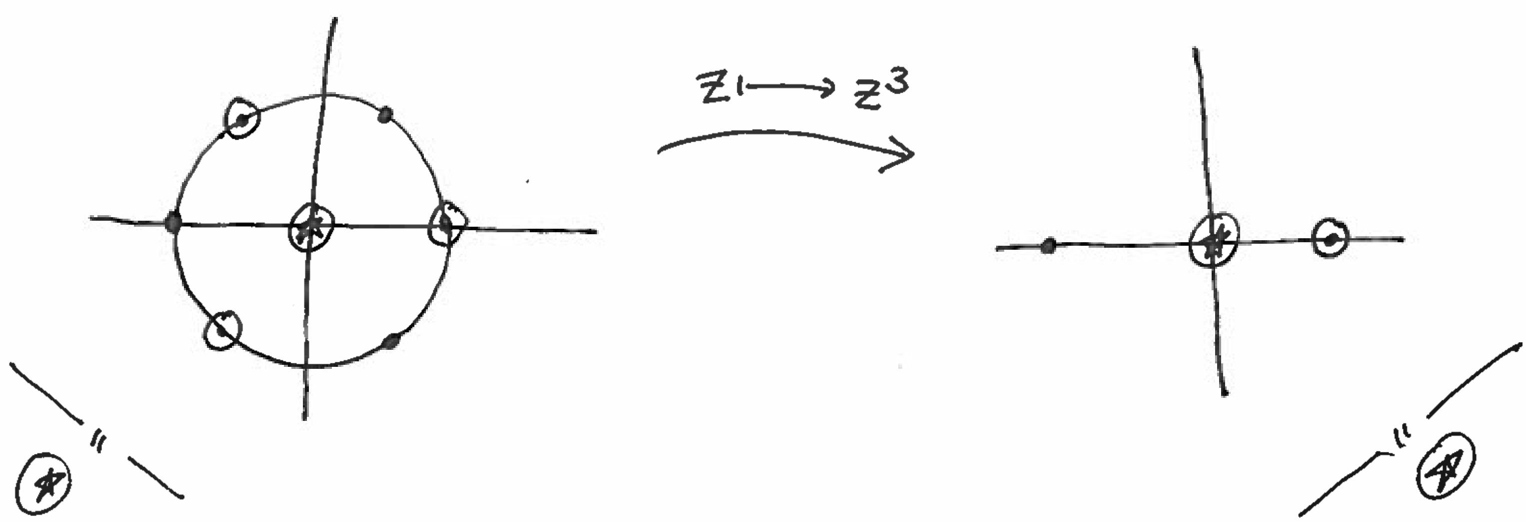
<u>Riemann Existence Thm:</u> (Special Case) There $\exists$ a rat'l function $\mathbb{P}^1_{\mathbb{C}} \to \mathbb{P}^1_{\mathbb{C}}$ which "matches" $\pi$ and so $G$ acts on $\mathbb{P}^1_{\mathbb{C}}$ by projective transformations.

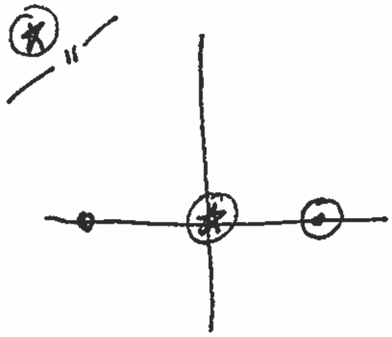
[This is the (complex) analysis hammer... ]

Think of $\mathbb{P}^1_{\mathbb{C}}$ as $\mathbb{C} \cup \{\infty\}$ which we identify with our picture  via sterographic projection



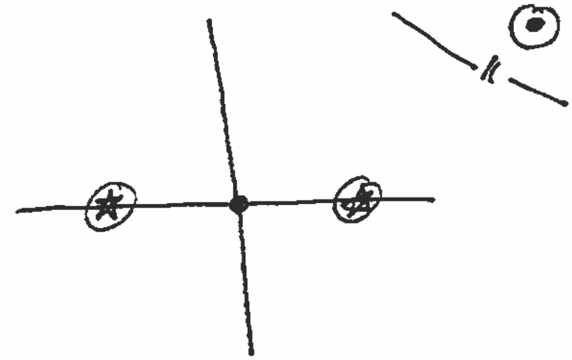

$$G = \langle b = (z \mapsto \zeta_3 z),\ a = (z \mapsto 1/z) \rangle$$





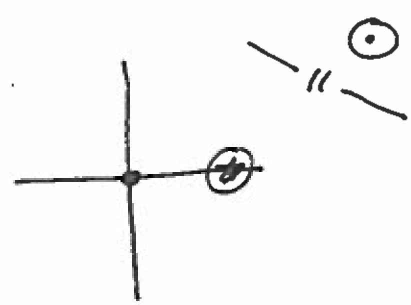
$$z \longmapsto \frac{z+1}{-z+1}$$

proj trans coor to $\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$



Composit is $h(z) = \left(\dfrac{z^3+1}{z^3-1}\right)^2$

$\downarrow \; z \longmapsto z^2$



Easy to check that if $\sigma \in G$ then $h \circ \sigma = h$ and that $h$ is the quotient map $\mathbb{P}^1_{\mathbb{C}} \to \mathbb{P}^1_{\mathbb{C}}/G$

From $\mathbb{P}^1_{\mathbb{C}} \xrightarrow{h} \mathbb{P}^1_{\mathbb{C}}$ we get $\mathbb{C}(z) \xleftarrow{h^*} \mathbb{C}(t)$

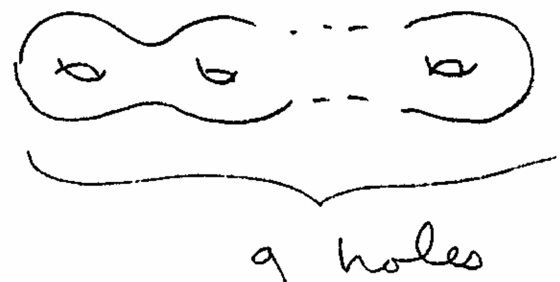
$$\left(\frac{z^3+1}{z^3-1}\right)^2 \longleftarrow t$$

Expanding, get $(t-1)z^6 + 2(t+1)z^3 + t = 0$.

That is, $\mathbb{C}(z)/\mathbb{C}(t) \cong \mathbb{C}(t)[u] \Big/ (t-1)u^6 + 2(t+1)u^3 + t$

As we have $G$ acting on $\mathbb{C}(z)$ fixing $\mathbb{C}(t)$, must have $[\mathbb{C}(z) : \mathbb{C}(t)] \geq 6$ and so is irreducible.

Notes: In general, the big field will not be $\cong \mathbb{C}(t)$. In fact, this is only poss when $G = \mathbb{Z}_K, D_K, T, O, I$
tet. oct. icos.

In general, $Y =$ 


and $|G| \leq 84(g-1)$.

When $g=3$, the most symmetries is 168. This is realized by the Klein quartic from HW.

Proof uses hyperbolic geometry, also needed for Wiles-Taylor proof of FLT...

The End