Lecture 17: Invertibility and isomorphisms

V, W vector spaces.

$I_V : V \longrightarrow V$ identity transformation; I_W similar.

$v \longmapsto v$

<u>Def</u>: Suppose $T: V \longrightarrow W$ is linear. A function $S: W \longrightarrow V$ is an <u>inverse</u> to T if $S \circ T = I_V$ and $T \circ S = I_W$.

<u>Thm</u>: Suppose $T: V \longrightarrow W$ is linear

1) T has an inverse if and only if it is 1-1 and onto.
2) If T has an inverse, it is unique, and denoted $T^{-1}: W \longrightarrow V$.
3) If T^{-1} exists, then it to is linear.

<u>Proof</u>: 1 and 2 are standard facts about functions between sets; see [Appendix B, FIS].

For 3, suppose $w_1, w_2 \in W$ and $c \in \mathbb{R}$. Let

v_1, v_2 be the unique elts in V with $T(v_i) = w_i$.

Then $T(cv_1 + v_2) = cT(v_1) + T(v_2) = cw_1 + w_2$.

[Thus $cv_1 + v_2$ is the vector in V which T takes to $cw_1 + w_2$.] Hence

$$T^{-1}(cw_1 + w_2) = cv_1 + v_2 = cT^{-1}(w_1) + T^{-1}(w_2).$$

So T^{-1} is linear as claimed. $\blacksquare$

Ex: $T : \mathbb{R}^2 \to \mathbb{R}^2$ rotation by $\pi/2$ counter clockwise.
$(x,y) \mapsto (-y, x)$

$S : \mathbb{R}^2 \to \mathbb{R}^2$ rotation by $\pi/2$ clockwise.
$(x,y) \mapsto (y, -x)$

$$T \circ S(x,y) = T(S(x,y)) = T(y, -x) = (x,y) = I_{\mathbb{R}^2}(x,y)$$

$$S \circ T(x,y) = S(T(x,y)) = S(-y, x) = (x,y) = I_{\mathbb{R}^2}(x,y).$$

If $\beta = \{e_1, e_2\}$, then

$$[T]_\beta = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \qquad [S]_\beta = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

<u>Thm</u>: Suppose $T: V \to W$ is linear and β is a basis for V. If T is invertible, then $\gamma = \{T(v) \mid v \in \beta\}$ is a basis for W.

<u>Cor</u>: If V is finite dim'l and $T: V \to W$ is invertible, then W is also finite-dim'l with $\dim(V) = \dim(W)$.

<u>Pf</u>: For easy of notation, I'll only do the case when β is finite, say $\beta = \{v_1, \dots, v_n\}$.

Set $w_i = T(v_i)$ so that $\gamma = \{w_1, \dots, w_n\}$.

<u>γ spans</u>: Let $w \in W$. There are unique scalars so that

$$T^{-1}(w) = a_1 v_1 + \cdots + a_n v_n.$$

Then

$$w = T(T^{-1}(w)) = T(a_1 v_1 + \cdots + a_n v_n) = \sum_{i=1}^{n} a_i w_i$$

as needed.

γ linearly indep: Suppose

$$a_1 w_1 + \cdots + a_n w_n = 0.$$

Then

$$0 = T^{-1}(0) = a_1 T^{-1}(w_1) + \cdots + a_n T^{-1}(w_n)$$
$$= a_1 v_1 + \cdots + a_n v_n$$

which implies all $a_i = 0$ as β is a basis. So γ is linearly independent.

Def: Vector spaces V and W are isomorphic when there exists an invertible linear trans $T: V \to W$. Such a T is called an isomorphism.

Ex: $T: \mathbb{R}^3 \to P_2(\mathbb{R})$ where $T(a_1, a_2, a_3) = a_1 + a_2 x + a_3 x^2$

Thm: Suppose that V is finite dim'l. Then some vector space W is isomorphic to V if and only if W is finite dim'l and $\dim W = \dim V$.

Proof: If V and W are isomorphic, then we've already shown $\dim V = \dim W$. For the converse, on the HW you'll show that any vector space of dim n is isomorphic to $\mathbb{R}^n$. As you can check, isomorphism is an equivalence relation, and so if $\dim V = \dim W = n$, we have V is isomorphic to W as both are isomorphic to $\mathbb{R}^n$. $\blacksquare$

Def: An $n \times n$ matrix A is invertible if there exists an $n \times n$ matrix B with $AB = BA = I_n$.

Ex: $A = \begin{pmatrix} 1 & 1 \\ 2 & 3 \end{pmatrix}$ $\quad B = \begin{pmatrix} 3 & -1 \\ -2 & 1 \end{pmatrix}$

$$AB = \begin{pmatrix} 1 & 1 \\ 2 & 3 \end{pmatrix}\begin{pmatrix} 3 & -1 \\ -2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$
$$BA = \begin{pmatrix} 3 & -1 \\ -2 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 2 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Note: When A has an inverse, it is unique since if C also has $AC = CA = I_n$ then $C = CI_n = CAB = B$. When it exists, the inverse of A is denoted A^{-1}.

$\beta \quad \gamma$ bases.

Connection: Suppose $T: V \longrightarrow W$ is an isomorphism between vector spaces of dim n. If

$A = [T]_\beta^\gamma$ and $B = [T^{-1}]_\gamma^\beta$ then $B = A^{-1}$

Proof: $$AB = [T]_\beta^\gamma [T^{-1}]_\gamma^\beta = [T \circ T^{-1}]_\gamma^\gamma$$
$$= [I_W]_\gamma^\gamma = I_n$$

Similarly
$$BA = [T^{-1}]_\gamma^\beta [T]_\beta^\gamma = [T^{-1} \circ T]_\beta^\beta$$
$$= [I_V]_\beta^\beta = I_n.$$