

# Lecture 21: Definition of the determinant

①

Last time:  $\det: M_{n \times n}(\mathbb{R}) \rightarrow \mathbb{R}$

Facts: ①  $\det(A) \neq 0 \iff A$  is invertible

②  $\det(AB) = \det(A) \det(B)$

③  $L_A: \mathbb{R}^n \rightarrow \mathbb{R}^n$  changes volumes by a factor of  $|\det(A)|$ .

④ Except when  $n=1$ ,  $\det$  is not linear.

For  $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$  have  $\det(A) = ad - bc$ .

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For an  $n \times n$  matrix  $A$ , let  $\tilde{A}_{ij}$  denote the  $(n-1) \times (n-1)$  matrix obtained by deleting the  $i$ th row and  $j$ th column of  $A$ .

Ex:  $A = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 0 & 2 \\ 2 & 1 & 1 \end{pmatrix}$      $\tilde{A}_{11} = \begin{pmatrix} 0 & 2 \\ 1 & 1 \end{pmatrix}$      $\tilde{A}_{23} = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$   
 $\tilde{A}_{33} = \begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix}$

Def: For  $A \in M_{1 \times 1}(\mathbb{R})$ , set  $\det(A) = A_{11}$ .

For  $A \in M_{n \times n}(\mathbb{R})$  with  $n > 1$ , inductively set

$$\det(A) = \sum_{j=1}^n (-1)^{j+1} A_{1j} \det(\tilde{A}_{1j})$$

Note: Matches old def for  $n=2$  as if  $A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$

we have

$$\det(A) = A_{11} \underbrace{\det(\tilde{A}_{11})}_{A_{22}} - A_{12} \underbrace{\det(\tilde{A}_{12})}_{A_{21}}$$

Ex:

$$\begin{aligned} \det \begin{pmatrix} 1 & 2 & 3 \\ 1 & 0 & 2 \\ 2 & 1 & 1 \end{pmatrix} &= 1 \cdot \det \begin{pmatrix} \tilde{A}_{11} \\ \tilde{A}_{12} \end{pmatrix} - 2 \cdot \det \begin{pmatrix} \tilde{A}_{12} \\ \tilde{A}_{13} \end{pmatrix} + 3 \cdot \det \begin{pmatrix} \tilde{A}_{13} \\ \tilde{A}_{12} \end{pmatrix} \\ &= 1 \cdot (-2) - 2 \cdot (-3) + 3 \cdot 1 = 7 \end{aligned}$$

Ex:

$$\begin{aligned} \det \begin{pmatrix} 4 & 3 & 2 & 1 \\ 3 & 1 & 2 & 3 \\ 2 & 1 & 0 & 2 \\ 1 & 2 & 1 & 1 \end{pmatrix} &= 4 \cdot \det(\tilde{A}_{11}) - 3 \cdot \det(\tilde{A}_{12}) + \dots \\ &= 4 \cdot 7 - \dots \\ &= 21. \end{aligned}$$

[Will develop better methods to compute det later using our trusty row ops; using the def requires  $n!$  multiplications...]

[To get to the properties mentioned at the beginning, must start small, those these are still useful...]

Thm: det is a linear function of the  $r^{\text{th}}$  row when all other rows are held fixed. In particular suppose  $A, B, C \in M_{n \times n}(\mathbb{R})$  are the same except in row  $r$  where  $a_r = b_r + k c_r$ . (here  $a_r$  is the  $r^{\text{th}}$  row of  $A$ , etc.). Then

$$\det A = \det B + k \det C.$$

[Query: How does this not violate ④ above?]

Proof: First suppose  $r = 1$ . Then

$$\det(A) = \sum_{j=1}^n (-1)^{j+1} A_{1j} \det(\tilde{A}_{1j})$$

$$= \sum_{j=1}^n (-1)^{j+1} (B_{ij} + k C_{ij}) \det(\tilde{A}_{ij})$$

↖ also =  $\tilde{B}_{ij}, \tilde{C}_{ij}$

$$= \sum_{j=1}^n (-1)^{j+1} B_{ij} \det(\tilde{B}_{ij}) + k \sum_{j=1}^n (-1)^{j+1} C_{ij} \det(\tilde{C}_{ij})$$

$$= \det(B) + k \det(C).$$

In general, we induct on  $n$ .

Base case:  $n=1$  where  $\det$  is actually linear.

Inductive step: Assume proven for  $M_{n \times n}(\mathbb{R})$ .

If  $r=1$  we are done by above, so assume  $r > 1$ . Then

$$\det(A) = \sum_{j=1}^{n+1} (-1)^{j+1} A_{ij} \det(\tilde{A}_{ij})$$

$= B_{ij} = C_{ij}$

↖ The same as  $\tilde{B}_{ij}$  and  $\tilde{C}_{ij}$  except in row  $(r-1)$ .

$$= \sum_{j=1}^{n+1} (-1)^{j+1} A_{ij} \left( \det(\tilde{B}_{ij}) + k \det(\tilde{C}_{ij}) \right)$$

$$= \det(B) + k \det(C).$$



Thm: Suppose  $A \in M_{n \times n}(\mathbb{R})$ . For any  $r$  with  $1 \leq r \leq n$  we have

(5)

$$\det(A) = \sum_{j=1}^n (-1)^{r+j} A_{rj} \det(\tilde{A}_{rj})$$

Lemma: Suppose  $B \in M_{n \times n}(\mathbb{R})$  where row  $r$  of  $B$  equal to  $e_j$ . Then  $\det(B) = (-1)^{r+j} \det(\tilde{B}_{rj})$

Note: When  $n=1$ , we have  $r=j=1$  and  $B=(1)$ .

So for this to hold, let's define  $\det(0 \times 0 \text{ matrix}) = 1$ .

Proof of theorem assuming the lemma:

Set  $B_j \in M_{n \times n}(\mathbb{R})$  to be  $A$  with the  $r^{\text{th}}$  row replaced by  $e_j$ . Thus

$$(r^{\text{th}} \text{ row of } A) = \sum_{j=1}^n A_{rj} (r^{\text{th}} \text{ row of } B_j)$$

By the first theorem, we have

$$\det(A) = \sum_{j=1}^n A_{rj} \det(B_j)$$

$$= \sum_{j=1}^n A_{rj} (-1)^{r+j} \det(\underbrace{(\tilde{B}_j)_{rj}}_{=\tilde{A}_{rj}})$$

⑥

as required. ◻

Ex:  $A = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 0 & 2 \\ 2 & 1 & 1 \end{pmatrix} \quad r=2$

$$\det(A) = -A_{21} \det(\tilde{A}_{21}) + A_{22} \det(\tilde{A}_{22}) - A_{23} \det(\tilde{A}_{23})$$

$$= (-1) \cdot \begin{vmatrix} 2 & 3 \\ 1 & 1 \end{vmatrix} + 0 \cdot \det(\tilde{A}_{22}) - 2 \cdot \begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix}$$

means "det of."

$$= (-1) \cdot (-1) - 2 \cdot (-3) = 7. \checkmark$$

Next time: Proof of the lemma.