Lecture 23: Properties of the determinant [§3.1 and 4.3]

Last time: $A \in M_{n\times n}(\mathbb{R})$

① $A \xleftrightarrow{R_r \leftrightarrow R_s} B \Rightarrow \det(B) = -\det(A)$

② $A \xrightarrow{cR_r} B \Rightarrow \det(B) = c\det(A)$

③ $A \xrightarrow{cR_s + R_r} B \Rightarrow \det(B) = \det(A)$

Today: $\det(AB) = \det(A)\det(B)$

[Strategy: Relate row ops to matrix mult. But first, here's one more easy consequence of what we learned last time.]

Recall that $\operatorname{rank}(A) = \dim(\operatorname{ColSp}(A)) \overset{\text{Thm}}{=} \dim(\operatorname{RowSp}(A))$

Thm: For $A \in M_{n\times n}(\mathbb{R})$, if $\underbrace{\operatorname{rank}(A) < n}_{\iff A \text{ not invertible}}$ then $\det(A) = 0$.

Pf: As $\operatorname{rank}(A) < n$, some row is a linear comb. of the others, say

$$a_r = c_1 a_1 + \cdots + c_{r-1} a_{r-1} + c_{r+1} a_{r+1} + \cdots + c_n a_n$$

where a_i is row i of A. If we row ops

$-c_i R_i + R_r$ for $1 \le i \le n$ and $i \ne r$, then we get a matrix B whose r^{th} row is 0.

Hence by last time, $\det(B) = 0$. By 3, we have $\det(A) = \det(B)$ and so $\det(A) = 0$ as required. ▨

Def: An $n \times n$ elementary matrix is the result of doing a single row operation to I_n.

Ex: 1
$$I_3 \xrightarrow{R_1 \leftrightarrow R_2} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$
$$I_4 \xrightarrow{R_1 \leftrightarrow R_4} \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

2
$$I_3 \xrightarrow{5R_2} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

3
$$I_3 \xrightarrow{5R_2 + R_1} \begin{pmatrix} 1 & 5 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad I_4 \xrightarrow{-3R_1 + R_4} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -3 & 0 & 0 & 1 \end{pmatrix}$$

[Note these are the identity matrix except in one position (2 and 3) or in four positions 1.]

Thm: Suppose E is the elementary matrix where $I_n \xrightarrow{R} E$. If $A \in M_{n \times m}(\mathbb{R})$, then $A \xrightarrow{R} EA$

Pf: On HW #7.

Ex:

$$\begin{array}{ll} & \quad E \qquad\quad A \\ R_1 \leftrightarrow R_2,\ n=2 & \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 3 & 4 \\ 1 & 2 \end{pmatrix} \\ 3R_1 & \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 3 & 6 \\ 3 & 4 \end{pmatrix} \\ -R_1 + R_2 & \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 2 & 2 \end{pmatrix} \end{array}$$

Thm: Every elementary matrix is invertible.

Pf: Suppose $I_n \xrightarrow{R} E$. Let R' be the row operation that reverses R, so that $A \xrightarrow{R} B \xrightarrow{R'} A$ for all $A \in M_{n\times n}(\mathbb{R})$. [Query: why does R' exist?]

Let E' be the elementary matrix associated to R'.

By the previous theorem, have

$$E'E = \text{result of doing } R' \text{ to } E = I_n$$

$$EE' = \text{result of doing } R \text{ to } E' = I_n$$

So E is invertible with inverse E'. $\blacksquare$

Thm: $A \in M_{n\times n}(\mathbb{R})$ is invertible if and only if it is the product of elementary matrices.

Pf: ($\Leftarrow$) If $A = E_1 E_2 \cdots E_\ell$ with E_k elementary, then each E_k is invertible and so

$$A^{-1} = (E_\ell)^{-1}(E_{\ell-1})^{-1} \cdots (E_1)^{-1}$$

by HW.

($\Rightarrow$) If A is invertible, then

$$(A \,\vdots\, I_n) \xrightarrow[\text{ops}]{\text{row}} (I_n \,\vdots\, A^{-1})$$

As each row op can be implemented by mult. by an elementary matrix, we have E_k where

$$E_\ell \cdots E_2 E_1 A = I_n$$

So

$$A = E_1^{-1} E_2^{-1} \cdots \cdots E_{\ell-1}^{-1} E_\ell^{-1}$$

As each E_k^{-1} is also elementary, we're done. ▨

Thm: $\det(AB) = \det(A)\det(B)$

Pf: If $\operatorname{rank}(AB) < n$ then $\det(AB) = 0$. Moreover, one of A and B must have rank $< n$ and so one of $\det(A)$ and $\det(B)$ is 0. So in this case

$\det(AB) = \det(A)\det(B)$ as needed.

So have reduced to the case where A, B, and AB all have rank n. In particular

$$A = E_1 E_2 \cdots E_\ell \quad \text{and} \quad B = E_{\ell+1} E_{\ell+2} \cdots E_m$$

where the E_K are elementary. The result now follows from:

<u>Claim</u>: Suppose $C = E_1' \cdots E_p'$ where E_k' are elementary. Then:

$$\det(C) = (-1)^{(\#\text{ of type } 1\ E_k')} \begin{pmatrix} \text{product of } c_k \\ \text{in all type } 2\ E_k' \end{pmatrix}$$

<u>Pf of Claim:</u> C is obtained from I_n by row ops $R_p', R_{p-1}', \ldots, R_1'$. Now $\det(I_n) = 1$, and we know only type 1 and 2 ops change the determinant, and they do so in a way that shows the claim. ▨

<u>Cor</u>: For $A \in M_{n\times n}(\mathbb{R})$, have $\det(A) \neq 0$ if and only if A is invertible.

<u>Pf</u>: If A is invertible, then $1 = \det(I_n)$ $= \det(AA^{-1}) = \det(A)\det(A^{-1}) \Rightarrow \det(A) \neq 0$.

In instead A is not invertible, then $\det(A) = 0$ by the first result of today. ▨