Lecture 28: Proof of the diagonalization criteria. ①

Last time: λ eigenvalue of $A \in M_{n \times n}(\mathbb{R})$.

Multiplicities:

Algebraic: # of times $(t-\lambda)$ divides the char poly of A

Geometric: $\dim E_\lambda$

Thm: $A \in M_{n \times n}(\mathbb{R})$ is diagonalizable if and only if

a) The char poly of A splits completely over $\mathbb{R}$

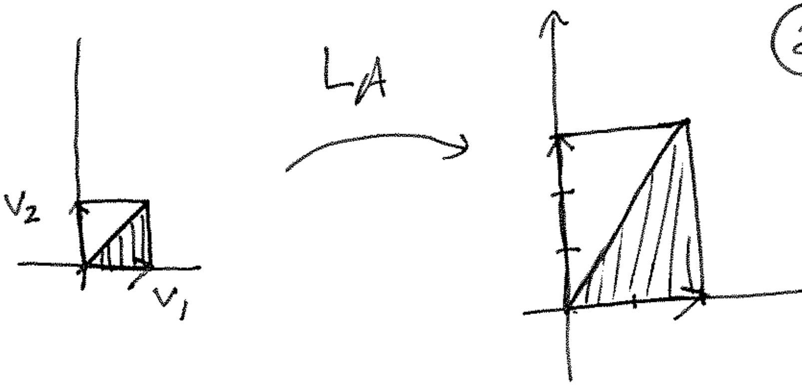
b) (alg. mult) = (geom mult) for all eigenvalues of A.

Lemma: Suppose $v_1, v_2, \ldots, v_k \in \mathbb{R}^n$ are eigenvectors of A corresponding to distinct eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_k$. Then $\{v_1, \ldots, v_k\}$ are linearly independent.

Moral: Can't create an eigenvector with eigenvalue λ from eigenvectors with other eigenvalues.

Ex: $A = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$

$v_1 = e_1 \quad v_2 = e_2$



$A(e_1 + e_2) = Ae_1 + Ae_2 = 2 \cdot e_1 + 3 \cdot e_2$

Proof of Lemma: Can assume $\lambda_k \neq 0$ for $k > 1$. Induct on k.

Base case: As v_1 is an eigenvector, it is nonzero and so $\{v_1\}$ is linearly independent.

Inductive Step: Assume $\{v_1, v_2, \ldots, v_{k-1}\}$ is linearly independent. Will prove by contradiction, so assume $\{v_1, \ldots, v_k\}$ is linearly dependent.

Then $v_k = a_1 v_1 + \cdots + a_{k-1} v_{k-1}$ for some $a_i \in \mathbb{R}$.

Now
$$Av_k = \lambda_k v_k = \sum_{i=1}^{k-1} (\lambda_k a_i) v_i$$

and
$$Av_k = \sum_{i=1}^{k-1} A(a_i v_i) = \sum_{i=1}^{k-1} (\lambda_i a_i) v_i$$

Now $Av_k \neq 0$ but we have two <u>distinct</u> ways of writing it as a linear combination of the linearly indep. set $\{v_1, \ldots, v_{k-1}\}$, which is impossible. So $\{v_1, \ldots, v_k\}$ is linearly independent, completing the induction. $\blacksquare$

<u>Lemma</u>: Suppose $\lambda_1, \ldots, \lambda_k$ are distinct eigenvalues for A. If $\beta_i \subseteq E_{\lambda_i}$ is lin. indep., then $\beta = \beta_1 \cup \cdots \cup \beta_k$ is linearly independent.

<u>Note</u>: $\beta_i \cap \beta_j = \emptyset$ for $i \neq j$ as if $v \neq 0$ then $Av = \lambda_i v$ and $Av = \lambda_j v \Rightarrow \lambda_i = \lambda_j$.

<u>Proof</u>: Suppose $\beta_i = \{v_1^i, v_2^i, \ldots, v_{d_i}^i\}$ and there are scalars a_j^i such that

$$\sum_{i=1}^{k} \underbrace{\sum_{j=1}^{d_i} a_j^i v_j^i}_{w_i} = 0$$

Each $w_i \in E_{\lambda_i}$ and is either an eigenvector for λ_i or is 0. By earlier lemma, can't have a linear dependence among eigenvectors with different eigenvalues, so all $w_i = 0$. As each β_i is linearly indep, conclude $a^i_j = 0$ for all i and j. So β is linearly independent. $\blacksquare$

Proof of Thm: ($\Leftarrow$) By (a) have

$$\text{char poly of } A = \pm(t-\lambda_1)^{m_1}(t-\lambda_2)^{m_2}\cdots(t-\lambda_k)^{m_k}$$

for distinct $\lambda_i \in \mathbb{R}$. Notice $\sum_{i=1}^{k} m_i = \deg(\text{char poly}) = n$. Let β_i be a basis for E_{λ_i}. By (b), know $\#\beta_i = \dim E_{\lambda_i} = m_i$. Set $\beta = \beta_1 \cup \cdots \cup \beta_k$. Now $\#\beta = \sum \#\beta_i = n$ and β is lin. indep by the lemma. So β is a basis of $\mathbb{R}^n$ consisting of eigenvectors for A, and so A is diagonalizable.

($\Longrightarrow$) Suppose A is diagonalizable. Showed last time that (a) follows, so let λ_i, m_i be as before. Set $d_i = \dim E_{\lambda_i}$ which we know satisfies $d_i \leq m_i$. Let β be a basis for $\mathbb{R}^n$ consisting of eigenvectors of A. Set $b_i = \#$ of v in β that are in E_{λ_i}. Know

$b_i \leq d_i \leq m_i$ and so

$$n = \#\beta = \sum b_i \leq \sum d_i \leq \sum m_i = n$$

Thus we must have $b_i = d_i = m_i$ for all i, proving (b).

QED.