Lecture 3: Subspaces (§1.3 of [FIS])

Previously on Math 416...

A vector space over $\mathbb{R}$ is a set V with two operations (vector addition and scalar mult) satisfying: (1-2) vec. addition is commutative and associtive.

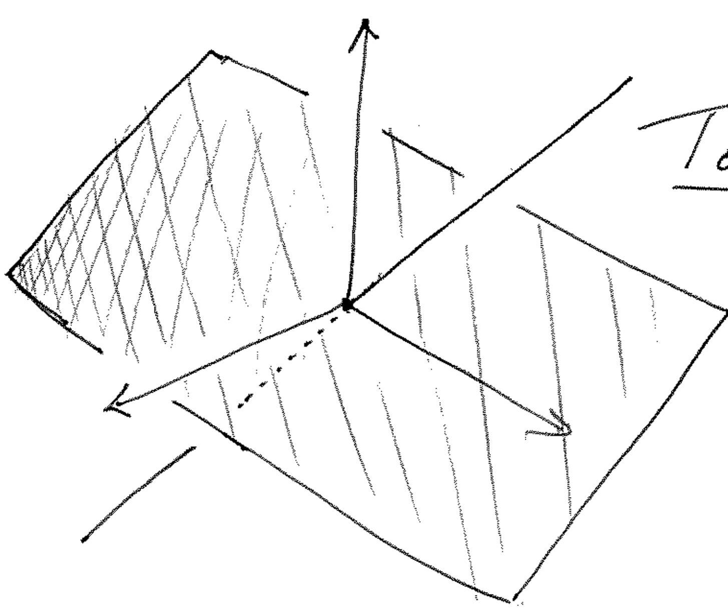
(3) There is a zero vector. (4) Additive inverses exist.

(5) $1v = v$ (6) scalar mult is assoc.

(7-8) Distributive properties.

Ex: $\mathbb{R}^n$, $\mathrm{Mat}_{m\times n}(\mathbb{R})$, spaces of functions...

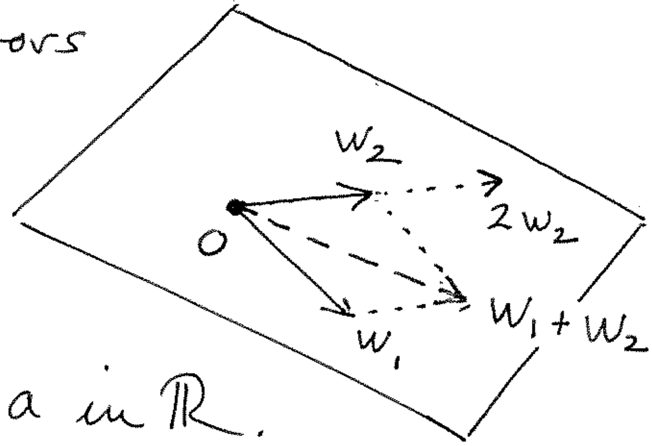
Back to $\mathbb{R}^3$: Other basic objects: lines and planes.



Today: Analog of such in a general vector space.

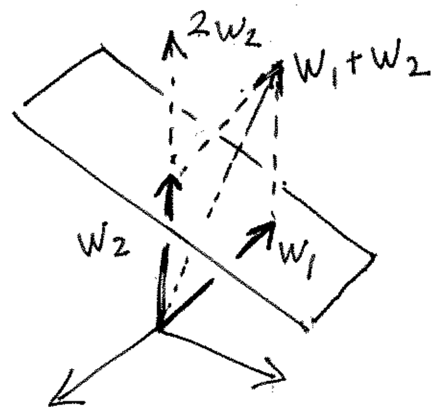
Suppose W is a plane in $\mathbb{R}^3$ containing 0, and w_1, w_2 are vectors in W. Then

$w_1 + w_2$ is also in W.



So is $a w_1$ for any a in $\mathbb{R}$.

Note: Important that W contains 0 here as otherwise these props need not hold.



Def: Suppose V is a vector space over $\mathbb{R}$. A subset W of V is a subspace if
(a) 0 is in W
(b) For all w_1, w_2 in W, the sum $w_1 + w_2$ is also in W.
(c) For all c in $\mathbb{R}$ and w in W, cw is also in W.

[Can replace (a) with requirement that W is nonempty.]

Ex: Some subspaces of $\mathbb{R}^3$:

1. $\mathbb{R}^3$
2. $\{0\}$
3. $\{(x,0,0) \text{ for } x \text{ in } \mathbb{R}\}$
4. $\{(x,-x,2x)\}$
5. $\{(x,y,0)\}$
6. $\{x+y+z=0\} = \{s(1,0,-1)+t(1,-1,0) \text{ for } s,t \text{ in } \mathbb{R}\}$

Ex: In any vector space V, the subsets $\{0\}$ and V are subspaces.

Thm: Suppose W is a subspace of a vector space V. Then W is itself a vector space under the two operations inherited from V.

Proof: First by requirements (b) and (c) we do have two ops taking values in W.

Of the 8 conditions, (1-2) and (5-8) are immediate from the fact that V itself is a vector space. Moreover, (3) follows from subspace cond. (a).

Finally, for 4 given w in W we know there is a v in V such that $v + w = 0$.

<u>Issue:</u> Does v have to be in W?

Yes, since we can take $v = (-1)w$ which is in W by c. <u>Check:</u> $v + (-1)v \overset{5}{=} 1v + (-1)v \overset{8}{=} (1-1)v$

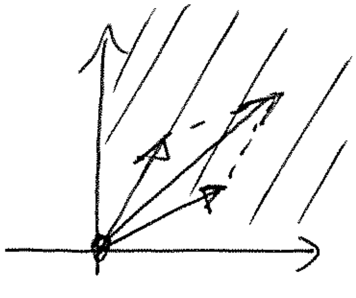
$$= 0v = 0$$

↑ Thm of last time.

So W with these ops satisfies 1-8 and so is a vector space. ∎

Proof end symbol (= Q.E.D.)

<u>Non-Ex:</u> $W = \{(w_1, w_2) \text{ with } w_i \geq 0\}$ in $\mathbb{R}^2$ is not a subspace.



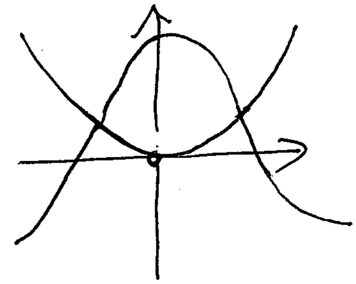
Satisfies a and b but not c.

In proof of thm, everything works except 4.

[Discuss difference with book's treatment of subspaces.]

Ex: $\mathcal{F} = \{$cont. fns $[-1,1]$ to $\mathbb{R}\}$

$W = \{f$ in $\mathcal{F}$ where $f(0) = 0\}$



So x^2 in W but $\cos x$ is not.

W is a subspace since

ⓐ The 0 in $\mathcal{F}$ is f_0 where $f_0(x) = 0$ for all x. which is in W.

ⓑ If f, g in W then $(f+g)(0) = f(0) + g(0) = 0 + 0 = 0$. So $f+g$ in W.

ⓒ If c in $\mathbb{R}$ and f in W then $(cf)(0) = cf(0) = 0$. So cf in W.

Non Ex: $\mathcal{F}$ same, $W = \{f$ in $\mathcal{F}$ where $f(0) = 1\}$

Fails all 3 req's!

Ex: A in $\mathrm{Mat}_{n\times n}(\mathbb{R})$

$$A = \begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{21} & A_{22} & & \\ \vdots & & \ddots & \\ A_{n1} & A_{n2} & \cdots & A_{nn} \end{pmatrix} = (A_{ij})$$

Transpose: A^t where $A^t_{ij} = A_{ji}$

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}^t = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix} \qquad \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}^t = \begin{pmatrix} 1 & 4 & 7 \\ 2 & 5 & 6 \\ 3 & 6 & 9 \end{pmatrix}$$

[Also works for non-square matrices.]

A matrix A in $\mathrm{Mat}_{n\times n}(\mathbb{R})$ is symmetric if $A = A^t$.

Ex: $\begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$ but neither of the two examples above.

Thm: The subset of symmetric matrices in $\mathrm{Mat}_{n\times n}(\mathbb{R})$ is a subspace.

<u>Proof</u>: The 0 in $Mat_{n\times n}(\mathbb{R})$ is $\begin{pmatrix} 0 & \cdots & 0 \\ & \ddots & \\ 0 & & 0 \end{pmatrix}$ which is symmetric so ⓐ holds.

For ⓑ and ⓒ, first show that for all A, B in $Mat_{n\times n}(\mathbb{R})$ and a, b in $\mathbb{R}$ one has

$$(aA + bB)^t = a(A^t) + b(B^t).$$

Now if A, B are sym, then

$$(A+B)^t = A^t + B^t = A + B$$

and so $A+B$ is also sym, proving ⓑ.

The argument for ⓒ is similar. $\blacksquare$

<u>Next time</u>: Linear combinations and linear equations