Lecture 37: Diagonalizing self-adjoint operators (§6.4)

Last time: A linear operator T on an inner product space is self-adjoint when $T^* = T$. A square matrix is self-adjoint when $A^* = A$.

Synonyms: Hermitian (field $= \mathbb{C}$), symmetric (field $= \mathbb{R}$)

Ex: Suppose W is a subspace of a finite dim'l inner product space V. Then orthogonal projection

$$\mathrm{proj}_W : V \to V \text{ is self-adjoint.}$$

Reason: For $x \in V$ and $w \in W$, note that

$\langle x, w \rangle = \langle \mathrm{proj}_W(x), w \rangle$ since

$x = \mathrm{proj}_W(x) + z$ with $z \in W^\perp$. Hence

for all $x, y \in V$ we have

$$\begin{aligned}\langle \mathrm{proj}_W(x), y \rangle &= \langle \mathrm{proj}_W(x), \mathrm{proj}_W(y) \rangle \\ &= \langle x, \mathrm{proj}_W(y) \rangle\end{aligned}$$

and so proj_W is self-adjoint.

Note: $R(\mathrm{proj}_W) = W$ and $N(\mathrm{proj}_W) = W^\perp$.

By the Dimension Thm, get $\dim W + \dim W^\perp = \dim V$. [Also $R^\perp = N$; will show this is a general property of normal ops on HW. On to today's big theorem.]

Thm: Suppose T is a self-adjoint operator on a finite dim'l inner product space V. Then V has an orthonormal basis β consisting of eigenvectors for T. In particular, T is diagonalizable.

Cor: If $A \in M_{n\times n}(\mathbb{R} \text{ or } \mathbb{C})$ is self-adjoint, then it is diagonalizable.

Lemmas from Last Time: a) Any eigenvalue of a self-adjoint T is real. b) Any self-adjoint operator on $V \neq \{0\}$ has an eigenvalue/eigenvector.

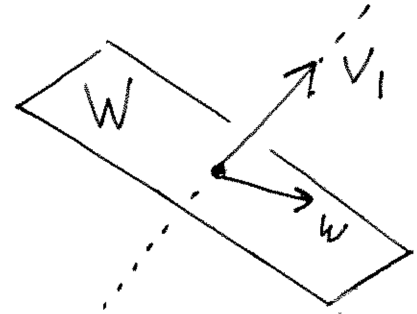
[Remind of connection between (a) and (b).]

<u>Proof of Thm</u>: We induct on $\dim(V)$.

If $\dim(V) = 1$, let v be an eigenvector of T given by the lemma. As $v \neq 0$, we have $\beta = \left\{ \frac{v}{\|v\|} \right\}$ is an orthonormal basis of V consisting of eigenvectors of T.

Now assume theorem holds whenever $\dim(V) \leq n-1$, and consider V of dim n. Let v_1 be a unit eigenvector of T given by the lemma. Consider,

$W = \{v_1\}^{\perp} = \mathrm{span}(v_1)^{\perp}$.



<u>Claim</u>: $T(W) \subseteq W$.

Reason: Suppose $w \in W$. Then $\langle T(w), v_1 \rangle$

$$\underset{\substack{\uparrow \\ \text{as } T \text{ self-adjoint}}}{=} \langle w, T(v_1) \rangle = \langle w, \lambda_1 v_1 \rangle = \overline{\lambda_1} \langle w, v_1 \rangle$$

$$= \overline{\lambda_1} \cdot 0 = 0 \quad \text{since } w \in \{v_1\}^{\perp}.$$

So $T(w)$ is also in $\{v_1\}^{\perp} = W$.

Since $W = \text{span}(v_1)^\perp$, have

$$\underbrace{\dim(\text{span}(v_1))}_{1} + \dim W = \dim V = n$$

and so $\dim W = n-1$.

Now W is also an inner product space (with the inner product inherited from V), and the restricted linear operator $T_W : W \longrightarrow W$ is still self-adjoint. By induction, there is an orthonormal basis $\{v_2, \dots, v_n\}$ of W consist. of eigenvectors of T_W. Then $\beta = \{v_1, v_2, \dots, v_n\}$ is an orthonormal set (as $\langle v_1, v_i \rangle = 0$ for $i \neq 1$ as $v_i \in \{v_1\}^\perp$) of eigenvectors of T. As any orthonormal set is linearly indep. and $\# \beta = \dim V$, we conclude that β is a basis for V, completing the induction. ∎

[One last special class of linear ops on inner product spaces.]

Suppose V is an inner product space. A linear op T on V is an <u>isometry</u> when

$$\langle T(x), T(y) \rangle = \langle x, y \rangle \text{ for all } x, y \in V.$$

Also called <u>orthogonal</u> when $\mathbb{F} = \mathbb{R}$ and
<u>unitary</u> when $\mathbb{F} = \mathbb{C}$.

[Reflect on the theme of preserving structure...]

<u>Ex</u>: $(\mathbb{R}^2, \text{dot})$: rotations, reflections
$(\mathbb{R}^n, \text{dot})$: rigid motion fixing 0.

<u>Non Ex</u>: $(\mathbb{R}^2, \text{dot})$: $T = L_A$ for $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.

Reason: $\langle T(e_2), T(e_2) \rangle = \langle (1,1), (1,1) \rangle = 2$

but $\langle e_2, e_2 \rangle = 1$.

$(V, \langle \, , \rangle)$: Any T with $\mathcal{N}(T) \neq \{0\}$.

<u>Thm</u>: A linear op T on V is an isometry if and only if $\|T(v)\| = \|v\|$ for all $v \in V$.

<u>Pf</u>: ($\Rightarrow$) Clear since $\|x\| = \sqrt{\langle x, x\rangle}$ by definition.

($\Leftarrow$) If fact, $\|\cdot\|$ determines $\langle \cdot, \cdot \rangle$. By expanding $\langle x+y, x+y\rangle$, get that

$$\mathrm{Re}(\langle x, y\rangle) = \frac{1}{2}\left(\|x+y\|^2 - \|x\|^2 - \|y\|^2\right)$$

↑real part

and if $\mathbb{F} = \mathbb{C}$ then

$$\mathrm{Im}(\langle x, y\rangle) = -\frac{1}{2}\left(\|ix+y\|^2 - \|x\|^2 - \|y\|^2\right)$$

Thus if T preserves $\|\cdot\|$, it also preserves $\langle \cdot, \cdot \rangle$. ▨