Lecture 38: Orthogonal and unitary operators (§6.5)

Convention: Today, V will always be a finite-dim'l inner product space over $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$.

Story so far: T linear op. on V.

$$\left.\begin{array}{l} \underline{\text{Normal}}: \; T \circ T^* = T^* \circ T \\ \underline{\text{Self-Adjoint}}: \; T^* = T \end{array}\right\} \quad \langle T(x), y \rangle = \langle x, T^*(y) \rangle$$

Isometry: $\langle T(x), T(y) \rangle = \langle x, y \rangle$ for all $x, y \in V$.
(aka orthogonal/unitary) [Recall examples...]

Thm: For a linear op T on V, the following are equivalent:

a) T is an isometry.

b) $\|T(x)\| = \|x\|$ for all $x \in V$.

c) $T \circ T^* = T^* \circ T = I_V$ $\left(\Rightarrow \begin{array}{l} T \text{ normal} \\ T^* = T^{-1} \end{array} \right)$

d) For every orthonormal basis β of V, the image $T(\beta)$ is also an orthonormal basis

e) For some orthonormal basis β of V, $T(\beta)$ is orthonormal.

<u>Proof:</u> Learned (a) $\iff$ (b) last time, and (d) $\Rightarrow$ (e) is clear.

<u>(a) $\Rightarrow$ (d)</u>: Suppose $\beta = \{u_1, \ldots, u_n\}$ and set $w_i = T(u_i)$. As T is an isometry, have $\langle w_i, w_j \rangle = \langle u_i, u_j \rangle$ and so $\gamma = \{w_1, \ldots, w_n\}$ is also orthonormal. Moreover, γ is a basis since $\#\gamma = \#\beta = \dim V$.

<u>(c) $\Rightarrow$ (b)</u>: $\|T(x)\|^2 = \langle T(x), T(x) \rangle = \langle x, \overbrace{T^* \circ T}^{I_V}(x) \rangle$

$$= \langle x, x \rangle = \|x\|^2 \text{ for all } x \in V.$$

<u>(e) $\Rightarrow$ (c)</u>: Suppose $\beta = \{u_1, \ldots, u_n\}$ is an orthonormal basis such that $\gamma = \{w_1, \ldots, w_n\}$ with $w_i = T(u_i)$ is also orthonormal. It suffices to show $T^* \circ T = I_V$ as then $T^* = T^{-1}$ and hence $T \circ T^* = I_V$ as well. Now set $v_i = T^* \circ T(u_i)$.

Then $\langle v_i, u_j \rangle = \langle T^*(T(u_i)), u_j \rangle$

$$= \langle T(u_i), T(u_j) \rangle = \langle w_i, w_j \rangle = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$$

Thus have $v_i = u_i$ for all i and so $T^* \circ T = I_V$. ▨

If T is an isometry of V and β an orthonormal basis for V, then by (c) we have

$$I_n = [I_V]_\beta = [T^* \circ T]_\beta = [T^*]_\beta \circ [T]_\beta$$

$$= ([T]_\beta)^* [T]_\beta$$

Setting $A = [T]_\beta$, have $A^*A = I_n$ and $A^{-1} = A^*$.

Def: A square matrix is unitary when $A^*A = I$.
It is orthogonal when $A^tA = I$.

So the matrix of an isometry with respect to an orthonormal basis is always unitary, and when $\mathbb{F} = \mathbb{R}$ it is also orthogonal.

Thm: Suppose $A \in M_{n\times n}(\mathbb{R})$ is orthogonal. Then L_A is an isometry of $(\mathbb{R}^n, \text{dot})$.

Note: Analog is true for $A \in M_{n\times n}(\mathbb{C})$ that are unitary.

Proof: Let $\beta = \{e_1, e_2, \ldots, e_n\}$ be the standard basis for $\mathbb{R}^n$, which is orthonormal. Set $a_i = L_A(e_i)$ $=$ i^{th} column of A. Set $G = A^tA = \begin{pmatrix} - & a_1 & - \\ & \vdots & \\ - & a_n & - \end{pmatrix} \begin{pmatrix} | & & | \\ a_1 & \cdots & a_n \\ | & & | \end{pmatrix}$ and note $G_{ij} = a_i \cdot a_j$. Since $G = I$, this means $\{a_1, \ldots, a_n\}$ is orthonormal and hence L_A is an isometry by (e). $\square$

Cor: For $A \in \mathrm{Mat}_{n \times n}(\mathbb{R})$, the following are equivalent:

i) A is orthogonal

ii) $A^t = A^{-1}$

iii) The columns of A are an orthonormal basis for $(\mathbb{R}^n, \mathrm{dot})$

iv) The rows of A ______ " ______

v) L_A is an isometry of $(\mathbb{R}^n, \mathrm{dot})$.

Proof: Exercise.

Restated Thm: Suppose $A \in M_{n \times n}(\mathbb{R})$ is symmetric. Then there is an orthogonal Q with $Q^tAQ = Q^{-1}AQ$ diagonal.

Operators in Quantum Mechanics:

$\mathcal{H}$ = Hilbert Space of the system = inner complex product space

<u>Ex 1</u>: Two particles, each of which is either "spin up" $\uparrow$ or "spin down" $\downarrow$.

$\mathcal{H}$ = 4 dim'l inner product space over $\mathbb{C}$ with orthonormal basis $\underbrace{\{e_{\uparrow\uparrow}, e_{\uparrow\downarrow}, e_{\downarrow\uparrow}, e_{\downarrow\downarrow}\}}_{\text{pure states.}}$

<u>Ex 2</u>: Single particle moving in one dimension

$$\mathcal{H} = L^2(\mathbb{R}) = \{f: \mathbb{R} \to \mathbb{C} \text{ "reasonably nice"}\}$$

$$\langle f, g \rangle = \int_{-\infty}^{\infty} f(x)\bar{g}(x)\, dx.$$

At time t, state of system is described by a <u>unit</u> vector $\psi_t \in \mathcal{H}$.

<u>Ex 1</u>: $\psi_0 = \frac{2}{\sqrt{6}} e_{\uparrow\uparrow} + \frac{i}{\sqrt{6}} e_{\uparrow\downarrow} - \frac{1}{\sqrt{6}} e_{\downarrow\downarrow}$ (superposition of pure states.)

Here, if measure system will find it in state $\uparrow\uparrow$ with probability $4/6$ and in state $\uparrow\downarrow$ with prob $1/6$.

Observables (position, momenta, energy, ...)

are self-adjoint operators A on $\mathcal{H}$

Ex 1: (1st particle is spin up) = projection onto span$\{e_{\uparrow\uparrow}, e_{\uparrow\downarrow}\}$

Expected values of an observable A is computed in terms of decomposition of ψ_t as a linear combination of eigenvectors of A (which is diagonalizable!)

The operator corresponding to the total energy of the system is the Hamiltonian H. The time evolution of the system is governed by Schrödinger's equation $i\frac{\partial}{\partial t}\psi_t = H\psi_t$

which implies $\psi_t = U_t\,\psi_0$ where U_t is the unitary transformation $U_t = \exp(-itH)$. Here, I am implicitly using the matrix exponential:

$$\exp(X) = \sum_{n=1}^{\infty} \frac{1}{n!} X^n \quad \text{for } X \in M_{n\times n}(\mathbb{C}).$$