Math 20: Midterm Exam 1

Exam: Friday March. 10 (starts 9:05 sharp to 10:00).
Review Session: TBA (prob monday night).
Extra office hours: Thursday 1:00 - 3:00 (in addition to my usual office hours, M 2:30-3:30, Tu 3:00-4:00, F 2:30-3:30). You can also arrange to meet with me at other times if these are not convenient. CA's will have additional office hours to be announced.
Problem sessions: continue at usual times.
HW #10 Due Friday March. 10. Do any 10 of the review problems below (On problems where there are several identical sub-problems, you need do only one). This HW will not be graded as usual, but you will simply receive 1 point for each problem turned in. The idea is that by choosing the problems yourself, you can concentrate on the areas that you don't fully understand.
Material Covered: Through Monday's lecture, including the material on the HW due Wed, March 8.

Review problems and course outline

- Solving Systems of linear equations: Section 1.2
  - Gauss-Jordan method and variants.
  - Geometric interpretation of a system of linear equations.
  - Section 1.2: #13, 15, 16, 23. Section 1 Supp. Exercises #6.
- Vectors in $\mathbb{R}^2$, $\mathbb{R}^3$ and $\mathbb{R}^n$: Sections 3.1-3, 3.5, 4.1
  - Arithmetic of vectors (inc. geometric interpretation of)
  - Dot product, angles, projection.
  - Lines and planes in $\mathbb{R}^3$.
  - Section 3.3: #6, 11
  - Section 3.5: #4(b), 5(c), 12(a), 15(a), 21, 24, 26, 29.
- Linear Transformations: Section 4.2
  - General functions from $\mathbb{R}^n$ to $\mathbb{R}^m$.
  - Linear Transformations: matrix associated to, composition of as matrix multiplication, geometric understanding of.
  - Section 4.2 #16(b-c), 17, 20.
  - Additional problems: R1, R2, R3. (see back)
- Matrices: Chapters 1 and 2.
  - Operations on.

  - Inverses of.
  - The determinant and its properties.
  - Section 1.5: #6(b-c), 7(c), Section 1.6: #5, Section 2.2: #4, 7, Section 2.3: #4, Section 2.4: #5, 7
  - Additional problems: R4. (see back)

- Subspaces of $\mathbb{R}^n$: Anton-Rorres 5.2-5.6
  - Definition. Examples of subspaces in $\mathbb{R}^2$ and $\mathbb{R}^3$.
  - Geometric understanding of what a subspace is.
  - Linear combinations of vectors. Span of a set of vectors.
  - Linear independence and bases. Finding bases. Coordinates of a vector in terms of a basis.
  - Subspaces associated to a matrix: nullspace, columnspace, rowspace. The fact that for a matrix A with n columns: $rank(A) + nullity(A) = n$
  - Review problems: Section 5.2: #1, 11. Section 5.3: #3, 5. Section 5.4: #1(a-b), 3(b-d), 7(b-c), 17, 20. Section 5.5: #6, 9, 11. Section 5.6: #2, 4, 12(a). Chapter 5 Supplementary exercises: 3(a), 5(a).

- Projection and regression: Anton-Rorres 6.4 and 9.3
  - Projection onto a subspace. Geometric meaning of.
  - Least-squares fitting and regression via projection.
  - Review Problems: Section 6.4 #4(b), 9(through part c). Section 9.3: #1, 3.

Additional review problems.

R1: Without using the formula in the book, find the matrix for the linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^2$ which acts by rotating 45 degrees clockwise.
R2: Find the matrix of a linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^2$ which collapses $\mathbb{R}^2$ to the line $y = -x$. Is the resulting matrix invertible? (Note: There is more than one correct answer for the first part).
R3: Draw the image of the integer grid under the linear transformation whose matrix is

$$A = \begin{pmatrix} -1 & 1 \\ 1 & 2 \end{pmatrix}.$$

R4: Suppose A is a two by two matrix which acts on $\mathbb{R}^2$ like this:

What is $|\det(A)|$?

Sample midterm

The following is last semester's first midterm together with part of last semesters second midterm (this semester, the first midterm is latter and so covers more). As such, it is longer than the actual midterm will be.

There are 100 points possible on this exam. Take care to note that problems are not weighted equally. Calculators, books, notes and suchlike aids to gracious living are *not* allowed. On the questions that ask "Why?" a long answer is not what I'm looking for – 1-2 sentances is what I want. You have 55 minutes. Show your work. Good luck!

1. Consider the vectors $\mathbf{v} = (1, 2, -1)$ and $\mathbf{w} = (1, 0, 1)$ in $\mathbb{R}^3$. Compute

   (a) $\mathbf{v} - 2\mathbf{w}$ **(4 points)**

   (b) The angle between $\mathbf{v}$ and $\mathbf{w}$. **(4 points)**

2. Consider the matrices

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \end{pmatrix}, B = \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{pmatrix}, C = \begin{pmatrix} 2 & 1 \\ 2 & 1 \end{pmatrix}, D = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 0 & 1 \\ 1 & 1 & -2 \end{pmatrix}.$$

Compute the following, when possible (when not possible, just answer "not possible") **(8 points each)**

(a) AB.

(b) BA.

(c) $A^T + 2B$.

(d) $\det(D)$.

3. Let P_1 be the plane in $\mathbb{R}^3$ defined by $x+y = 1$, and P_2 the plane defined by $2x+y+3z = 2$.

(a) Find normal vectors for P_1 and P_2. **(5 points)**

(b) These two planes are not parallel. Why? **(5 points)**

(c) Parameterize the line which is the intersection of P_1 and P_2. **(10 points)**

4. Let $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation with matrix $A = \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}$, and $S\colon \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation with matrix $B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. **(5 points each)**

(a) Draw what happens to the unit square with vertices $(0,0), (1,0), (0,1)$, an $(1,1)$ under the transformation T.

(b) Based on your answer to (a), it the matrix A invertible or not? Why?

(c) Find the matrix associated with the linear transformation $T \circ S$.

5. **(10 points)** Find the matrix of the linear transformation $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ which acts as shown:

6. Consider the line segment L in $\mathbb{R}^3$ joining the points $P = (2, 0, -2)$ and $Q = (4, 2, 0)$.

   (a) The midpoint M of L is the point half-way between P and Q. Find M. **(5 points)**

   (b) Find the equation of the plane which intersects L in its midpoint M and which is orthogonal to L. **(10 points)**

7. Are the following 3 vectors in $\mathbb{R}^2$ linearly independant? Provide a one sentence justification of your answer. $v_1 = (2, 3), v_2 = (-2, 3), v_3 = (1, -1)$.

8. Consider the matrix

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 4 \end{pmatrix}$$

   (a) Find a basis for the nullspace of A. Also find the nullity of A, that is, the dimension of the nullspace.

   (b) Find the rank of A, that is, the dimension of the columnspace of A.

9. Answer the following question True or False.

   (a) If two vectors $\mathbf{v}$ and $\mathbf{w}$ in $\mathbb{R}^2$ are linearly independent the $\{\mathbf{v}, \mathbf{w}\}$ is a basis for $\mathbb{R}^2$.

   (b) If $\mathbf{v}$ and $\mathbf{w}$ are vectors in $\mathbb{R}^3$ then the subspace span$\{\mathbf{v}, \mathbf{w}\}$ is a plane.