Final Exam for Math 20, Spring Semester

Exam: Friday, May 26 at 2:15pm in Emerson 101 (exam will be 3 hours long).

Review Session: Friday, May 12 9:00am in Sci Center 309 (note that's in the morning, not the evening). The review session will focus on linear algebra and eigen-stuff.

Office hours: During the first week of reading (May 8-12) my office hours will be MWTh 2:00-3:30. **During the second week of reading, May 15-19, I will be out of town and unavailable to answer questions.** Justin and Patrick will have office hours during that week (to be announced). You may wish to take this into account and start studying early. My office hours will resume on Sat, May 20. The schedule will be: Sat, Monday, Wednesday, Thursday 2:00-4:00, Tuesday 10:00-12:00. Except for the week I am out of town, I'm also happy to meet with you by appointment. To get an appointment, send me mail (nathand@math) or stop by my office; I'm usually around and I'll probably even be able to talk to you then. My office is SC 334.

Material Covered, etc.: The final exam is comprehensive. There will be more questions on the material covered after the second midterm than is proportional to its percentage of the class. The material was about 20% of the class, and its proportion on the final will be about 35%.

The final exam will be like the two preceding exams, which, in turn, were very much like the homework problems. **However,** the questions on the final will be longer, more involved, and more difficult than the ones on the midterm exams, as you will have a longer time to work on each problem. Attached is last semester's final exam, which I view as about the right level of difficulty.

A good way to study for the exam is to work lots of review problems, though you should also review your notes and the corresponding sections of the text. You should also look at your old HW and make sure you can now do any problems you missed the first time round.

Good Luck!

Review problems and course outline

Note: The outline below is the essentially the same as previous handouts, but I've updated and expanded the lists of review problems. Also attached to this handout is a list of review problems (F1-11, R1-44) from previous final exams. **Warning: these questions are from exams not written by me and your final exam will differ.** There are also some additional problems labeled N1-6, as well as last semester's final exam (L1-11). Solutions to (almost all of) the review problems are posted on the course web site: http://www.math.harvard.edu/~nathand/

- Solving Systems of linear equations: AR Section 1.2
  - Gauss-Jordan method and variants.

  - Geometric interpretation of a system of linear equations.
  - Section 1.2: #13, 15, 23. Section 1 Supp. Exercises #7. R3, L1.

- Vectors in $\mathbb{R}^2$, $\mathbb{R}^3$ and $\mathbb{R}^n$: AR Sections 3.1-3, 3.5, 4.1
  - Arithmetic of vectors (inc. geometric interpretation of)
  - Dot product, angles, projection.
  - Lines and planes in $\mathbb{R}^3$.
  - Section 3.3: #6, 11, 13
  - Section 3.5: #4(a), 5, 12(b), 15(a), 20, 24, 26, 29.
  - F3, R2, R24, L5.

- Linear Transformations: AR Section 4.2
  - General functions from $\mathbb{R}^n$ to $\mathbb{R}^m$.
  - Linear Transformations: matrix associated to, composition of as matrix multiplication, geometric understanding of.
  - Section 4.2 #16(b-c), 17, 20.
  - Additional problems: N1(a), N2, N3.
  - R4, R30, L4, L11.

- Matrices: AR Chapters 1 and 2.
  - Operations on.
  - Inverses of.
  - The determinant and its properties.
  - Section 1.5: #6(b-c), 7(c), Section 1.6: #5, Section 2.2: #4, 7, Section 2.3: #4, Section 2.4: #5, 7
  - Additional problems: N4.

- Subspaces of $\mathbb{R}^n$: Anton-Rorres 5.2-5.6
  - Definition. Examples of subspaces in $\mathbb{R}^2$ and $\mathbb{R}^3$.
  - Geometric understanding of what a subspace is.
  - Linear combinations of vectors. Span of a set of vectors.
  - Linear independence and bases. Finding bases. Coordinates of a vector in terms of a basis.
  - Subspaces associated to a matrix: nullspace, columnspace, rowspace. The fact that for a matrix A with n columns: $rank(A) + nullity(A) = n$
  - Review problems: Section 5.2: #1, 11. Section 5.3: #2, 5, 18. Section 5.4: #1(a-b), 3(b-d), 7(b-c), 17, 18, 20. Section 5.5: #6, 9, 11. Section 5.6: #2, 4, 12(a). Chapter 5 Supplementary exercises: 3(a), 5(a).

- F5, R10, L2, L4.

- Projection and regression: Anton-Rorres 6.4 and 9.3
  - Projection onto a subspace. Geometric meaning of.
  - Least-squares fitting and regression via projection.
  - Review Problems: Section 6.4 #4(b), 9(through part c). Section 9.3: #1, 3.
  - F9, R38, L3.
- Functions of several variables: Supplement Ch. 11.
  - Functions of two variables: graphs, slices, level curves, contour diagrams.
  - Graphs, contour diagrams of linear functions.
  - Cobb-Douglas production functions.
  - Functions of 3-variables. Level surfaces.
  - Review Problems: 11.3: #17, 11.4: #34, 11.5: #17.
- Derivatives of functions of several variables: Supplement Ch. 13.
  - Partial derivatives.
  - Local linearity (differentiability) of functions of 2 variables. Normals and tangent planes to a graph. Approximation by linear functions.
  - Directional derivatives.
  - The gradient.
  - Chain rule.
  - Second order partial derivatives
  - Review problems: Section 13.3 #8 Section 13.4 #7 Section 13.5 #34, #36, Section 13.6 #4, 22. Section 13.7 #26. Chapter 13 Review problems: #41.
  - F37, F10, F12, R1, R12, R35, R36, L5.
- Optimization problems: Supplement Ch. 14
  - Local and global extrema, critical points, second derivative test.
  - Unconstrained optimization.
  - Constrained optimization: Lagrange multipliers.
  - Review problems: Section 14.2 #5. Chapter 14 Review #5,#7.
  - F8, R6, R7, R11, R32, R33, R34, L6, L7.
- Linear Programming: AR 11.3
  - Review Problems: Section 11.3 #2, 7

- Eigenvalues and Eigenvectors AR 7.1-2
  - Finding eigenvalues, characteristic polynomial.
  - Finding eigenvectors/bases for eigenspaces.
  - Diagonalizing matrices, applications to understanding powers of matrices.
  - Diagonalization as changing coordinates. (AR 8.4)
  - Review Problems: Section 7.1: #4-5-6(c), 13(c). Section 7.2: 13, 20. Section 8.4: 9. Extra Problems N1(b), R14, R26, L9, L10, L11.
- Dynamical Systems:
  - Discrete Dynamical Systems:
    - Markov Chains (AR 11.6). When they converge. Applying. Problems: F4, R8, R16.
    - Leslie Models (AR 11.18). When they converge: Perron-Frobenius Theorem. Applying. Problems: N5, N6.
  - Continuous Dynamical Systems AR 9.1:
    - Solving linear first order systems via diagonalization. Problems: 9.1 2, 6.
  - Converting between continuous and discrete systems.
  - Additional Problems: R28.

Additional review problems N1-N6.

N1: Let T be the linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^2$ which acts by rotating 45 degrees counterclockwise. Find the matrix for T in terms of (a) the standard basis, (b) the basis $\{v_1, v_2\}$ where $v_1 = (1, 1)$, and $v_2 = (-1, 0)$.

N2: Find the matrix of a linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^2$ which collapses $\mathbb{R}^2$ to the line $y = 2x$. Is the resulting matrix invertible? (Note: There is more than one correct answer for the first part).

N3: Draw the image of the integer grid under the linear transformation whose matrix is

$$A = \begin{pmatrix} 1 & 2 \\ 1 & -1 \end{pmatrix}.$$

N4: Suppose A is a two by two matrix which acts on $\mathbb{R}^2$ like this:

What is $|\det(A)|$?

N5: There are three interconnected rooms in a sleazy apartment. There are many flies buzzing about. During a long weekend, we find that after each hour $1/5$ of the flies in the living room will have wandered into the kitchen and $1/5$ into the bedroom. Of the flies that were in the bedroom, $1/3$ have migrated into the kitchen and $1/6$ into the living room. Of the flies in the kitchen the previous hour, $1/3$ are now in the bedroom and $1/4$ are in the living room. Assume in addition, that 10% of the flies die each hour, and new flies are born in the kitchen at a rate equal to 30% of the flies currently in the kitchen. Assume that the flies continue to exhibit this same behavior each hour. What happens to the total population of flies in the apartment over time? Does the population of flies stabilize? What are limiting ratios of the populations of flies in the various rooms?

N6: A bird species has a maximum life span of 3 years. On average, each pair of birds in their first year will produce two offspring. A typical sample of 8 birds in their 2nd year will produce a total of 16 offspring. After their second year, birds produce no more offspring. Only 50% of birds in their first year will survive to their second year, and only 20% of birds in their second year survive to their third year. Survival rates do not depend on gender. Describe how this bird population evolves over time. In particular, describe whether the population remains stable, decreases, or increases and at what rate. Furthermore, if possible describe the relative proportions of each age group after a number of years have passed. Use this information to compute amount (in the limit) each bird in their first or second year needs to pay into social security in order to pay for the retirement of the birds in their 3rd year, assuming each retired bird is entitled to $100/year.

Last Semester's final exam

1. **(20 points)** Consider the linear system

$$\begin{array}{rcrcrcr} 2x & + & ky & + & kz & = & 0 \\ x & + & ky & + & z & = & -1 \\ 2x & - & ky & + & (k-2)z & = & 2 \end{array}$$

   When does the system have a) no solutions, b) a unique solution, c) a parameter set of solutions, d) a two parameter set of solutions, e) a three parameter set of solutions? Note: Not all of a-e) necessarily occur.

2. Consider the matrix

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

   (a) Find a basis for the nullspace of A. Also find the the nullity of A, the dimension of the nullspace. **(8 points)**

   (b) Find the rank of A, and find a basis for the column space of A. **(6 points)**

3. **(10 points)** Use the method least squares (regression) to find the best fit of the form $y = a + bx + cx^2$ for the data:

x	0	1	2	3
y	2	2	1	3

   . Leave your answer in the form

   (vector involving a,b,c) = product of matrices, vectors

   but **do not multiply out matrices, take inverses, etc.**.

4. Let $\mathbf{v}_1 = (1, 1)$, $\mathbf{v}_2 = (1, -1)$, $\mathbf{w} = (5, -1)$, and $\mathcal{B}$ be the basis $\{\mathbf{v}_1, \mathbf{v}_2\}$.

   (a) **(5 points)** Find the coordinates of $\mathbf{w}$ with respect to the basis $\mathcal{B}$.

   (b) Let T be the transformation from $\mathbb{R}^2$ to $\mathbb{R}^2$ which is rotation by 90 degrees counter-clockwise. Find the matrix for T in terms of:

      i. The standard basis for $\mathbb{R}^2$. **(8 points)**

      ii. The basis $\mathcal{B}$. **(7 points)**

5. Consider the two surfaces in $\mathbb{R}^3$ defined by:

$$\begin{array}{ll} S_1: & x^2 + y^2 + z^2 = 3 \\ S_2: & 2x^2 - 8x + 2y^2 - 8y + z^2 - 10z = -33. \end{array}$$

   The point $\mathbf{p} = (1, 1, 1)$ lies on S_1, and the point $\mathbf{q} = (1, 1, 3)$ lies on S_2.

   (a) Find the equation for the tangent plane P_1 to S_1 at $\mathbf{p}$. Find equation for the tangent plane P_2 to S_2 at $\mathbf{q}$. **(10 points)**

   (b) The two planes you got in part (a) should be parallel. Why do the two equations define parallel planes? **(5 points)**

   (c) Consider the line segment L joining the points $\mathbf{p}$ and $\mathbf{q}$. Is this the shortest line segment joining the planes P_1 and P_2? **(5 points)**

6. Consider the function $f(x, y) = x^3 + y^3 + 3xy + 1/8$.

   (a) Determine all the local maxima, minima, and saddle points. Are the local extrema also global extrema? **(10 points)**

7. Let $f(x, y) = x^2 + xy$ and $g(x, y) = 2x^2 + 2xy + y^2$. Find the global min and max of f subject to the constraint $g(x, y) \leq 2$. **(22 points)**

8. Answer the following questions True or False. You do *not* need to justify your answers. Below, f is a function from $\mathbb{R}^2$ to $\mathbb{R}$. **(3 points each)**

   (a) Any 2×2 matrix is diagonalizable.

   (b) The directional derivative $f_{\mathbf{v}}(a, b)$ is a number.

   (c) If A is an $m \times n$ matrix (m rows and n columns), then A defines a linear transformations **from** $\mathbb{R}^m$ **to** $\mathbb{R}^n$.

   (d) The following 2 vectors are linearly independent in $\mathbb{R}^3$: $(1, 0, 1)$, $(0, 3, 5)$.

   (e) The following 3 vectors are linearly independent in $\mathbb{R}^2$: $(1, 3)$, $(2, -1)$, $(5, 7)$.

   (f) The vector $\operatorname{grad} f$ is always tangent to the level curves of f.

   (g) The set of all vectors in $\mathbb{R}^3$ of the form $(a, b, 1)$ is a subspace of $\mathbb{R}^3$.

   (h) The only subspaces of $\mathbb{R}^3$ are: the trivial subspace $\{\mathbf{0}\}$, lines through $\mathbf{0}$, and all of $\mathbb{R}^3$.

9. Let $A = \begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix}$. Find

   (a) The eigenvalues and eigenvectors of A. **(7 points)**

   (b) Use this information to compute A^{16}. **(8 points)**

10. The island of Madagascar is home to almost all of the worlds lemurs, a family of primitive non-ape primates. Divide Madagascar into 2 regions, the North and the South, along the Mania river. Suppose that in any given year, 1/3 of the lemurs living in the North move to the South, and 1/6 of those in the South move to the North.

    (a) Introduce the state vector $\mathbf{x}_k = \begin{bmatrix} n_k \\ s_k \end{bmatrix}$, where n_k is the number of lemurs living in the North in year k and, s_k is the number living in the South. Find the matrix A such that $\mathbf{x}_{k+1} = A\mathbf{x}_k$. **(5 points)**

    (b) Suppose the initial population is 12,000 lemurs, with one half living in the North and other half in the South. What happens to the population in the long term? How many lemurs live where? **(10 points)**

    (c) Now suppose in addition that in the North deaths exceed births and the population shrinks by 50 percent each year. Also, suppose that in the South, the population grows by 50 percent each year. Find a new matrix B so that with this new info, $\mathbf{x}_{k+1} = B\mathbf{x}_k$. Assume that the change in population levels happens *before* everyone moves around each year. **(10 points)**

(d) A wise old lemur tells you that the eigenvalues/eigenvectors of your new matrix B are:

$$\lambda_1 = 0.2899, v_1 = \begin{bmatrix} -5.7604 \\ 1 \end{bmatrix} \quad \text{and} \quad \lambda_2 = 1.2934, v_2 = \begin{bmatrix} 0.260399 \\ 1 \end{bmatrix}$$

What happens to the lemur population over time? Do they go extinct or flourish? At what rate does the population increase/decrease each year in the long term? **(5 points)**

11. **(10 points)** Consider the linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^2$ which does:

Let A be the matrix (in the standard basis) for the T. From the picture, what are the eigenvalues and eigenvectors of A? What is $\det(A)$?

1 Additional Review Problems F1-F12 and R1-R44

These are courtesy of Robert Winters, and come from past Math20 final exams. **Warning: these questions are from exams not written by me and your final exam will differ.** Not all these problems are relevant to our course.