Math 277: Exercises for weeks 1-4

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Note: These are just informal and not to be turned in. The difficulty varies widely. These problems will be updated regularly and can be downloaded from the course web page.

Introduction

1. Fix $n \geq 4$. Prove that any finitely presented group is π_1 of some n-manifold.
2. Prove that for a closed 3-manifold, $\pi_1 M$ trivial implies that M is homotopy equivalent to S^3.
3. Let M and N be hyperbolic n-manifolds. Prove that $\pi_1 M \cong \pi_1 N$ implies that M and N are homotopy equivalent.
4. Let Σ be surface of genus ≥ 2 with some fixed hyperbolic metric. Prove the following:
   (a) Any isotopy class of simple closed curves in Σ contains a unique geodesic.
   (b) Show that any two simple closed geodesics:
      i. are either the same or meet transversely.
      ii. meet in a minimal number of points (for simple closed curves in their isotopy classes).

Topological Foundations

Examples

1. Let K be a simplicial complex created by taking a bunch of 3-simplices and gluing their two dimensional faces together so that every face of every 3-simplex is glued to something. Prove that K is a 3-manifold if and only if the Euler characteristic of K is 0.
2. Prove that in an orientable 3-manifold, an embedded surface is 2-sided if and only if it is orientable.
3. Prove that any properly embedded disc in a 3-manifold is 2-sided.
4. Prove that the homeomorphism type of an orientable handlebody is determined by the number of handles. (Extra credit: What about non-orientable handlebodies?).
5. Prove that $L(1, n) \cong S^3$ and $L(2, 1) = \mathbb{R}\,\mathrm{P}^2$.
6. Produce an explicit homotopy equivalence between $L(7, 1)$ and $L(7, 2)$.
7. Find a Heegaard diagram for the 3-torus which minimizes the genus of the splitting.
8. Suppose K is a knot in S^3. Let $M = S^3 \setminus N(K)$. Consider Dehn surgery on K: create a closed 3-manifold by attaching the boundary of $D^2 \times S^1$ to the boundary of M via some homeomorphism f. Prove that the homeomorphism type of the resulting manifold depends only on the isotopy class of $f(D^2 \times \{\mathrm{pt}\})$.

9. Show that the two descriptions (via identifying the faces of a dodecahedron and via Dehn surgery on the trefoil) of the Poincaré homology sphere are the same.

10. Prove that there is a 3-manifold which cannot be obtained by Dehn surgery on S^3.

11. **(Skip for now)** Let L be the minimally twisted 5-chain link in S^3, and let M be $S^3 \setminus N(L)$. Convince yourself that the mapping class group of M is $\mathbb{Z}/2\mathbb{Z} \times S_5$. Hint: Look at $N(L)$ and quotient out S^3 by rotating through the axis of the chain. Each of the solid tori become balls, and the axis of the chain becomes a bunch of arcs that connect these balls. Thinking of this as a graph in S^3, this graph is the 1-skeleton of the triangulation that S^3 has when we view S^3 as the boundary of the 4-simplex.

Connected sum decomposition

1. Let M be a 3-manifold and $p\colon \tilde{M} \to M$ a covering map. Show that if $\tilde{M}$ is irreducible so is M. Since $\mathbb{R}^3$ and S^3 are irreducible, this gives lots of examples of irreducible 3-manifolds: Lens spaces, T^3 (or any Euclidean 3-manifold), any hyperbolic 3-manifold, unit tangent bundle to a hyperbolic surface.

2. Prove the smooth Jordan Curve Theorem: Every smoothly embedded circle in $\mathbb{R}^2$ bounds an embedded disc.

3. Use the method of Alexander's theorem to show that every torus embedded in S^3 bounds a solid torus $D^2 \times S^1$ on one side or the other. Give an example where it does so on only one side.

4. Classify the non-orientable 3-manifolds which are prime but not irreducible.

Incompressible surfaces and the loop theorem

1. Poincaré duality for 3-manifolds. Let M be a closed, orientable 3-manifold. The only interesting case of Poincaré duality for 3-manifolds is that $H^1(M, \mathbb{Z})$ is isomorphic to $H_2(M, \mathbb{Z})$. Fill in the following outline for a geometric proof (all (co)homology has coeffs in $\mathbb{Z}$):

   (a) Prove that any class x in $H_1(M)$ can be represented by an oriented embedded circle.

   (b) Prove that any class y in $H_2(M)$ can be represented by an oriented embedded surface (if you get stuck, this is Hatcher's Lemma 2.6).

   (c) Prove that $H^1(M)$ is naturally isomorphic to $H_1(M)^* = \mathrm{Hom}(H_1(M), \mathbb{Z})$.

   (d) There is a bilinear pairing $H_1(M) \otimes H_2(M) \to \mathbb{Z}$, namely the intersection (cap) product. If x is represented by an embedded circle and y is represented by an embedded surface with x and y intersecting transversely, this is just the number of times x crosses y, counted with signs. This gives a map from $H_2(M) \to H_1(M)^*$. Show that this map is injective.

   (e) Show that the map $H_2(M) \to H_1(M)^*$ is surjective: A class x^* in $H^1(M)$ gives a homotopy classes of maps $f\colon M \to S^1$ such that f^* of the fund. class of S^1 is x^*. Make f smooth and take the inverse image of a regular value to get a surface S in M. The class of S in $H_2(M)$ has image x^*.

   This proves that $H^1(M, \mathbb{Z})$ is isomorphic to $H_2(M, \mathbb{Z})$.

For extra credit, also do the case with where M has boundary: $H^1(M)$ iso. to $H_2(M, \partial M)$.

2. Let M be an orientable 3-manifold which contains a closed surface with odd Euler characteristic. Show that if F is such a surface with maximal odd Euler characteristic, then F is weakly incompressible.

3. Prove that the lens space $L(2k, p)$ contains a closed surface F with Euler characteristic $2 - k$. Also, prove that $L(2k, p)$ does not contain an embedded $\mathbb{R}\,\mathrm{P}^2$. Together with the preceding exercise, this gives many examples of weakly incompressible surfaces that are not strongly incompressible.

4. If M^3 is closed and non-orientable, prove M contains a strongly incompressible surface.

5. Let M be an irreducible 3-manifold with a fixed triangulation. If S is a (weakly) incompressible surface in M, show that S can be isotoped to be normal.

6. Later in the course, we will study Haken manifolds, those which contain incompressible surfaces. The presence of these surfaces (which are encoded in the fundamental group), allow one to prove theorems such as:

   Thm: If M is an irreducible 3-manifold homotopy equivalent to the 3-torus T^3 then M is homeo to T^3.

   Prove the following: If M is a prime 3-manifold which is homotopy equiv to $S^2 \times S^1$ prove that M is $S^2 \times S^1$. Hint: Rep the gen of $H_2(M)$ by an embedded incompressible surface, and show that surface must be a sphere.

   Extra credit: Prove the theorem for T^3.

Universal covers and contractible 3-manifolds

1. Let $W = S^3 \setminus X$ be the Whitehead manifold, where X is the intersection of the family of solid tori defined in class (see also Rolfsen).

   (a) Prove that X is connected.

   (b) Prove that X is not locally connected.

   (c) Prove that W not homeomorphic to $\mathbb{R}^3$.

   (d) Prove that $W \times \mathbb{R}$ is homeomorphic to $\mathbb{R}^4$.

2. A non-compact 3-manifold M is "simply connected at infinity" if the following holds. For every compact K in M there exists a compact set L containing K such that the inclusion $M \setminus L \hookrightarrow M \setminus K$ induces the zero homomorphism on π_1. It is worth mentioning that if M is the universal cover of a compact 3-manifold N then M being simply connected at infinity is really a property of "coarse geometry" of the group $\pi_1(N)$. We will talk more about "coarse geometry" later in the term.

   (a) Prove that $\mathbb{R}^3$ is simply connected at infinity.

   (b) Prove that the Whitehead manifold is not.

(c) Use the Loop Theorem to prove that if M is an irreducible non-compact 3-manifold which is simply connected at infinity then M is homeomorphic to $\mathbb{R}^3$. (Hint: first prove that an ascending union of 3-balls is $\mathbb{R}^3$.) Note: This result is special to dimension 3—in high dimensions there are manifolds which are simply connected at infinity but which are not $\mathbb{R}^n$.

3. Consider the following conjecture of Barry Mazur: Let M be a closed 3-manifold. Let $\widetilde{M}$ be its universal cover. Compactify $\widetilde{M}$ to get $\bar{M}$ by adding one point at infinity for each end of $\widetilde{M}$. Then $\bar{M}$ is S^3.

   Prove that this conjecture implies both the Poincaré conjecture, and the conjecture that an irreducible 3-manifold with infinite π_1 has universal cover $\mathbb{R}^3$.

The sphere theorem, irreduciblity of covers and minimal surfaces

1. Some references for minimal surfaces:

   Hass, Joel; Scott, Peter. The existence of least area surfaces in 3-manifolds. Trans. Amer. Math. Soc. 310 (1988), no. 1, 87–114.

   Jaco, William; Rubinstein, J. Hyam. PL minimal surfaces in 3-manifolds. J. Differential Geom. 27 (1988), no. 3, 493–524.

2. Let S be an embedded separating sphere in M. Prove that S is trivial in π_2 if and only if S bounds a homotopy ball. (A homotopy ball as a compact contractible 3-manifold with boundary. It is easy to show that such a manifold has boundary a 2-sphere).

3. Give an example of a *non-orientable* closed 3-manifold which is irreducible but which has a cover which is reducible.

4. Let G a finite group which is the fundamental group of some M^3. Prove that $H^*(G, \mathbb{Z})$ is periodic of period 4.

5. Prove that in a compact orientable 3-manifold M there exists a finite collection of embedded 2-spheres generating $\pi_2 M$ as a $\pi_1 M$ module.