Math 277: Topology and Geometry of 3-manifolds

Introduction

- This class is about compact 3-manifolds, for example: S^3, $T^3 = S^1 \times S^1 \times S^1$, $S^2 \times S^1$, or the unit tangent bundle to a surface.

  Fundamental Goal: Classify all compact 3-manifolds. What does classify mean? The ideal classification is that of surfaces:

  Theorem: Every compact connected 2-manifold without boundary is homeomorphic (or diffeomorphic) to one of the following:

  - (Orientable) The sphere, the torus or the connected sum of tori.
  - (Nonorientable) The projective plane, or a connected sum of proj. planes.

  The homeomorphism type of a surface is completely determined by its orientablity and its Euler characteristic (in other words, its homology) which are easily computable from, say, a triangulation.

  The classification of 3-manifolds may or may not be possible (though it probably is). This contrasts with dimensions ≥ 4 where classification of manifolds is impossible.

- The reason classification is impossible in high dimensions is group theoretic. Finitely presented groups can't be classified in any reasonable sense, and for any fixed $n \geq 4$ any finitely presented group is π_1 of some n-manifold. (Proof: For a f.p. group G build a finite 2-complex K with $\pi_1(K) = G$. Now as $n+1 \geq 5$, can embed K in $\mathbb{R}^{n+1}$. Let M be the boundary of a regular nbhd of K. Then M is a closed n-manifold and, since $n \geq 4$, $\pi_1 M = \pi_1 K = G$.) A reason finitely presented groups can't be classified is that there is no algorithm which can decide if two finitely presented groups are isomorphic. In fact, there is no algorithm to decide if a finitely presented group has any of the following properties: trivial, finite, free, nilpotent or simple.

- This doesn't mean you can't say anything about high-dimensional manifolds—in fact high-dimensional topology ($n \geq 5$) is far better understood than low-dimensional topology ($n = 3, 4$), once you mod out by the fact that it's impossible. In other words, fix some homotopy type K to get rid of the group theory and look at

  $$\{(M, f) \mid M \text{ is an } n\text{-manifold, } f\colon\ M \to K \text{ is a homotopy equivalence.}\}.$$

  moded out by homeomorphism (or if you're studying smooth manifolds, diffeomorphism). Often this set can be calculated with homotopy-theoretic methods (stable homotopy groups of spheres, L-groups, surgery exact sequences...).

- One of the most basic questions in 3-dimensions is unknown:

  Poincaré conjecture: Let M be a compact 3-manifold without boundary with $\pi_1 M$ trivial. Then M is homeomorphic to S^3.

  For a 3-manifold, $\pi_1 M$ trivial is equivalent to M homotopy equivalent to S^3. So you have the generalization:

Gen. Poincaré conjecture: Let M be a compact n-manifold without boundary homotopy equivalent to S^n. Then M is homeomorphic to S^n.

In 1960 Smale proved this was true in dimensions $n \geq 5$. If you replace homeomorphism by diffeomorphism the Generalized Poincaré Conjecture becomes false. For instance, Milnor showed that S^7 has 28 distinct differentiable structures (in general, you can calculate the number of smooth structures on S^n using stable homotopy groups of spheres). In dim 4, Freedman proved the gen. Poincaré around 1980.

- **Geometry:**

  Every surface has a metric of constant curvature. Often these metrics are useful for solving purely topological problems. As a toy example, let Σ be surface of genus ≥ 2 with some fixed hyperbolic metric. Any isotopy class of simple closed curves in Σ contains a unique geodesic. If we want to study isotopy classes of curves, it is convenient to look at the geodesic representatives since any two geodesics loops:

  - either the same or meet transversely.
  - meet in a minimal number of points (for their isotopy classes).

  Here's a couple of group-theoretic statements about $G = \pi_1(\Sigma)$ whose proofs use the fact Σ has a hyperbolic metric:

  1. G is residually finite, that is, the intersection of all its finite-index subgroups is the identity subgroup.
  2. G is subgroup separable, aka LERF. This means that given a subgroup H of G and an element $g \in G - H$ there exists a *finite-index* subgroup H' containing H with $g \notin H'$. The proof works by building the surface out of right-angled pentagons and looking at the induced tiling of $\mathbb{H}^2$.

- It would be nice if all manifolds had metrics of constant curvature, but in higher dimensions, very few manifolds do. The reason for this is that any n-manifold M with a constant curvature metric is a quotient of one of S^n, $\mathbb{E}^n$ or $\mathbb{H}^n$ by a group of isometries. So, for example, $\pi_2(M) = 0$ and hence e.g. $S^2 \times S^2$ or $\mathbb{C}\,\mathrm{P}^2$ don't have such metrics. Also, because $\pi_1(M)$ is a lattice in a Lie group, $\pi_1(M)$ has solvable word problem. However, many finitely presented groups do not have solvable word problem.

  Could generalize constant curvature to locally homogenous metrics, but still have solvable word problem.

  Around 1980 the theory of 3-manifolds was revolutionized by Thurston's realization that most 3-manifolds *should* have locally homogenous metrics:

  Geometrization Conjecture: Any compact 3-manifold can be cut into pieces along spheres and tori so that each piece can be given one of the 8 geometric structures: S^3, $\mathbb{E}^3$, $\mathbb{H}^3$, $S^2 \times \mathbb{R}$, $\mathbb{H}^2 \times \mathbb{R}$, Nil, Sol, $\widetilde{\mathrm{SL}_2\mathbb{R}}$.

  As in dimension 2, the generic case is $\mathbb{H}^3$. If true, this conjecture would be a big step toward classifying 3-manifolds. For instance, it implies that any 3-manifold group has solvable word problem, and is residually finite.

- In dimension 2 a surface other than S^2 or $\mathbb{R}\,\mathrm{P}^2$ has many constant curvature metrics. It is easy to see that the torus has a 2-dimensional space of flat metrics. For a surface of genus $g \geq 2$, the dimension of the space of hyperbolic metrics, up to isometry, is $6g - 6$. In dimension 3, the same flexibility is true for some geometries like $\mathbb{E}^3$ but in the generic case of $\mathbb{H}^3$ we have:

  Mostow Rigidity: Let M, N be compact hyperbolic n-manifolds with $n \geq 3$. Then if $\pi_1(M)$ is isomorphic to $\pi_1(N)$ then M and N are isometric.

  So for a hyperbolic 3-manifold, geometric invariants such as volume, length of shortest geodesic, or eigenvalues of the Lapacian are actually topological invariants. Dimension 3 is the unique dimension where topology and geometry more or less coincide.

Outline

- **Topological Foundations:** Weeks 1-4. Follows Hatcher.
  - Fundamental Goal: Classify compact 3-manifolds. Typical is the Poincaré conjecture: *Let M be a compact 3-manifold without boundary with $\pi_1(M) = 1$. Then M is homeomorphic to S^3.* Whether this is true is still unknown after a 100 years.
  - Examples: Triangulations, Heegaard splittings, and Dehn surgery.
  - Categories: Smooth, PL, and Top.
  - Connected sum decomposition.
    - Definitions and examples. Statement of decomposition theorem.
    - Every smooth S^2 in $\mathbb{R}^3$ bounds a ball.
    - Combinatorial minimal surfaces (aka normal surfaces).
    - Proof of theorem.
    - How this allows us to avoid the Poincaré conjecture, much of the time.
  - Homotopy to geometry (more normal surfaces)
    - Loop Theorem.
      - Incompressible surfaces.
    - Sphere Theorem: If M is a 3-manifold and $\pi_2(M) \neq 0$ then there is an *embedded* 2-sphere which is non-trivial in $\pi_2(M)$.
    - Consequences: Many 3-manifolds are $K(\pi, 1)$'s.
  - Normal surfaces and Algorithms for 3-manifolds.
- **The Geometry of 3-manifolds:** Weeks 4-6. Follows Bonahon, Scott.
  - Overview, dimension 2, dimensions > 3. Why dimension 3 is so special.
  - The eight 3-dimensional geometries.
  - Seifert fibered spaces.
  - Hyperbolic 3-manifolds.
  - JSJ decomposition theorem (decomposition along tori).

    - Special case of knot complements in S^3.
  - Thurston's Geometrization Conjecture.
  - Consequences of having a geometric structure: properties of the fundamental group.

- **Haken 3-manifolds:** Weeks 6-9. Follows Aitchison and Rubinstein.
  - Definition and examples. Proof that if M is an irreducible 3-manifold with $H_1(M, \mathbb{R}) \neq 0$ then M is Haken.
  - Haken hierarchies, or why Haken 3-manifolds are like surfaces. This leads to a sort of induction for Haken manifolds, which allows most major questions about them to be answered.
  - Topological rigidity for Haken 3-manifolds.
    - The Borel Conjecture: If two n-manifolds which are both $K(\pi, 1)$'s are homotopy equivalent, then they are homeomorphic.
    - Waldhausen's theorem: Any two Haken 3-manifolds which are homotopy equivalent are homeomorphic.
    - Comparison to Mostow Rigidity.
  - Weak Geometrization for Haken 3-manifolds.
    - Large scale geometry of groups and spaces (after Gromov).
    - Negative curvature in the large, word hyperbolic groups.
    - Isoperimetric inequalities.
    - Easy proof that an atoroidal Haken 3-manifold has word hyperbolic fundamental group. (Thurston proved the stronger statement that such manifolds have hyperbolic metrics, i.e. Riemannian metrics of constant curvature -1. The proof of Thurston's theorem is a lot more difficult.)
    - Properties of word hyperbolic groups: solving the word problem.

- **Consequences of Geometrization for Haken manifolds:** Weeks 10-13.
  - Knot Theory:
    - Computing hyperbolic structures in practice (SnapPea).
    - Solving the homeomorphism problem for hyperbolic 3-manifolds with cusps.
    - Combined, get practical way to decide if two knots in S^3 are the same.
  - The Smith Conjecture: Let f be a diffeomorphism of S^3 of finite order (that is, f^n is the identity map for some n). If f has a fixed point, then the fixed point set is a knot (embedded circle). The Smith conjecture asserts that this knot is unknoted, and f is conjugate to a rotation in $\mathbf{O}(4)$.
  - Dehn Surgery:
    - Examples and general problems.
    - The Cyclic Surgery Theorem.