Surfaces in finite covers of 3-manifolds:

The Virtual Haken Conjecture

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This talk available at http://dunfield.info/

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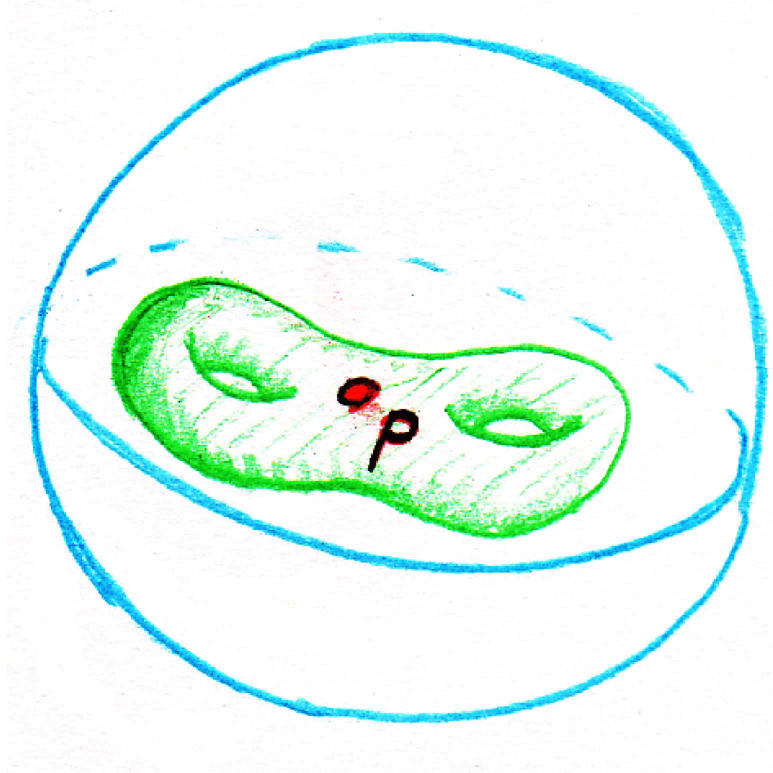
Natural place to start: studying surfaces Σ^2 in M^3. Need to ignore things like:



Convention: All manifolds are orientable.

Def. *A surface $\Sigma \neq S^2$ embedded in M^3 is incompressible if $\pi_1(\Sigma) \to \pi_1(M)$ is 1-1.*

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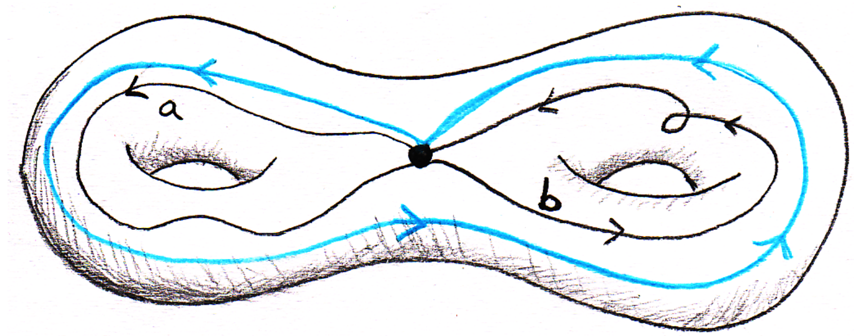

Ex. $\pi_1(S^3) = 1$.
$\pi_1(T^3) = \mathbb{Z}^3$, where $T = S^1 \times S^1 \times S^1 = \mathbb{R}^3/\mathbb{Z}^3$.
$\pi_1(W) =$
$\langle a, b \mid a^2b^2a^2b^{-1}ab^{-1} = b^2a^2b^2a^{-1}ba^{-1} = 1 \rangle$.

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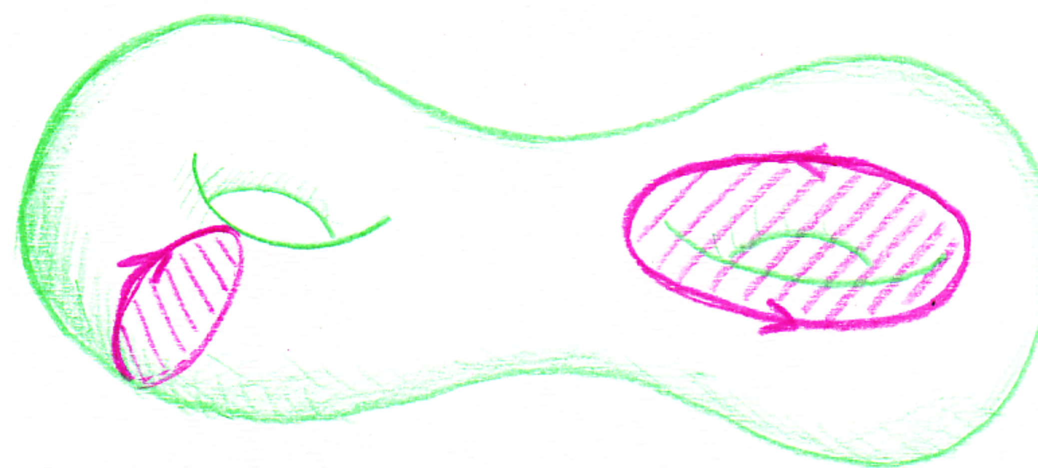

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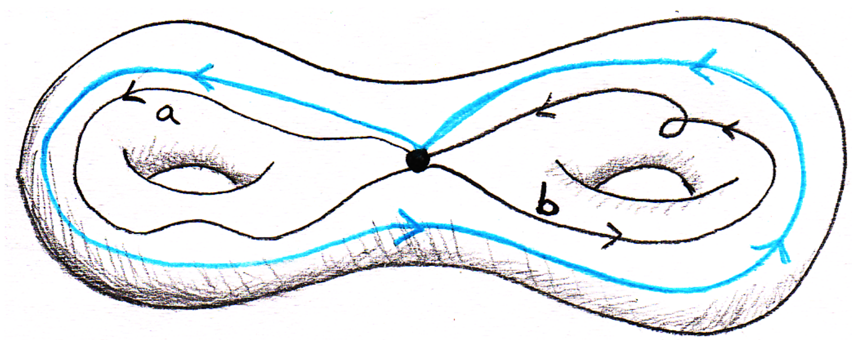

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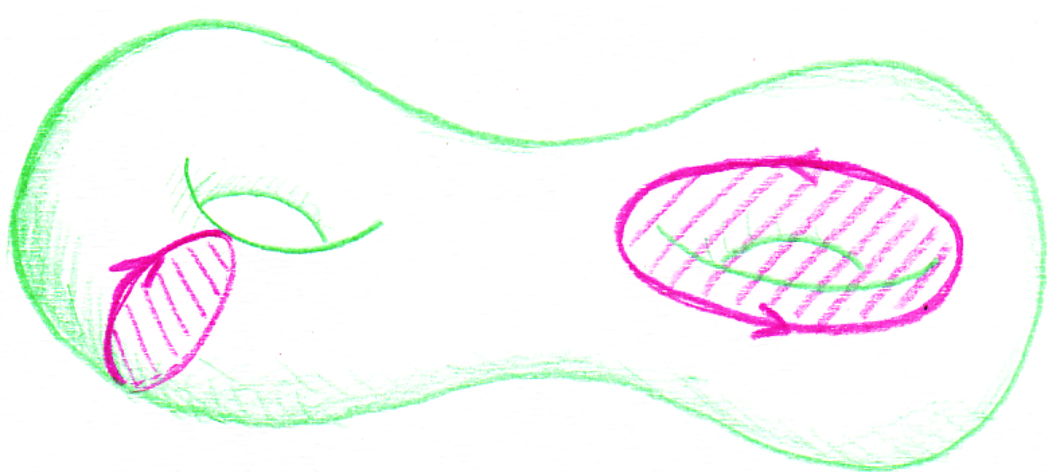


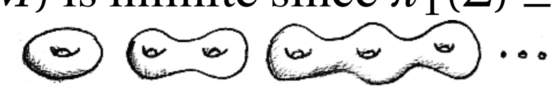
Incompressible: For $\Sigma = S^1 \times S^1 \times \{\text{pt}\} \subset T^3$, the map on π_1 is: $\mathbb{Z} \oplus \mathbb{Z} \hookrightarrow \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$.

Similarly, $\Sigma \times \{\text{pt}\}$ is an incompressible surface in $M^3 = \Sigma \times S^1$.

Def. *A compact M^3 is Haken if it is irreducible and contains an incompressible surface.*

Irreducible: Every embedded S^2 bounds a ball, that is, M is not a connected sum.

An arbitrary M^3 is of the form $M_1 \# M_2 \# \cdots \# M_n$ where the M_k can't be further decomposed.

If M is Haken, then $\pi_1(M)$ is infinite since $\pi_1(\Sigma) \leq \pi_1(M)$ and Σ is among: 

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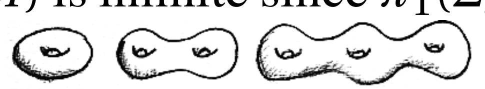
If M is Haken, then $\pi_1(M)$ is infinite since $\pi_1(\Sigma) \leq \pi_1(M)$ and Σ is among:  ...

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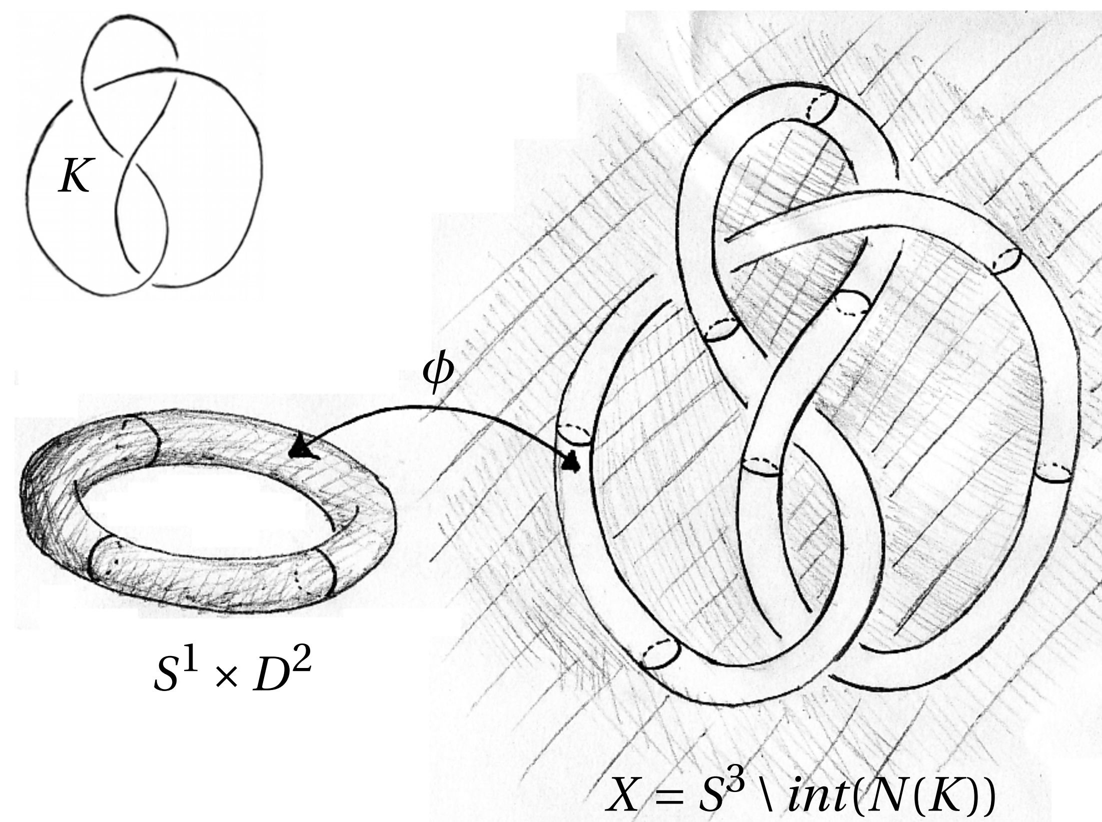

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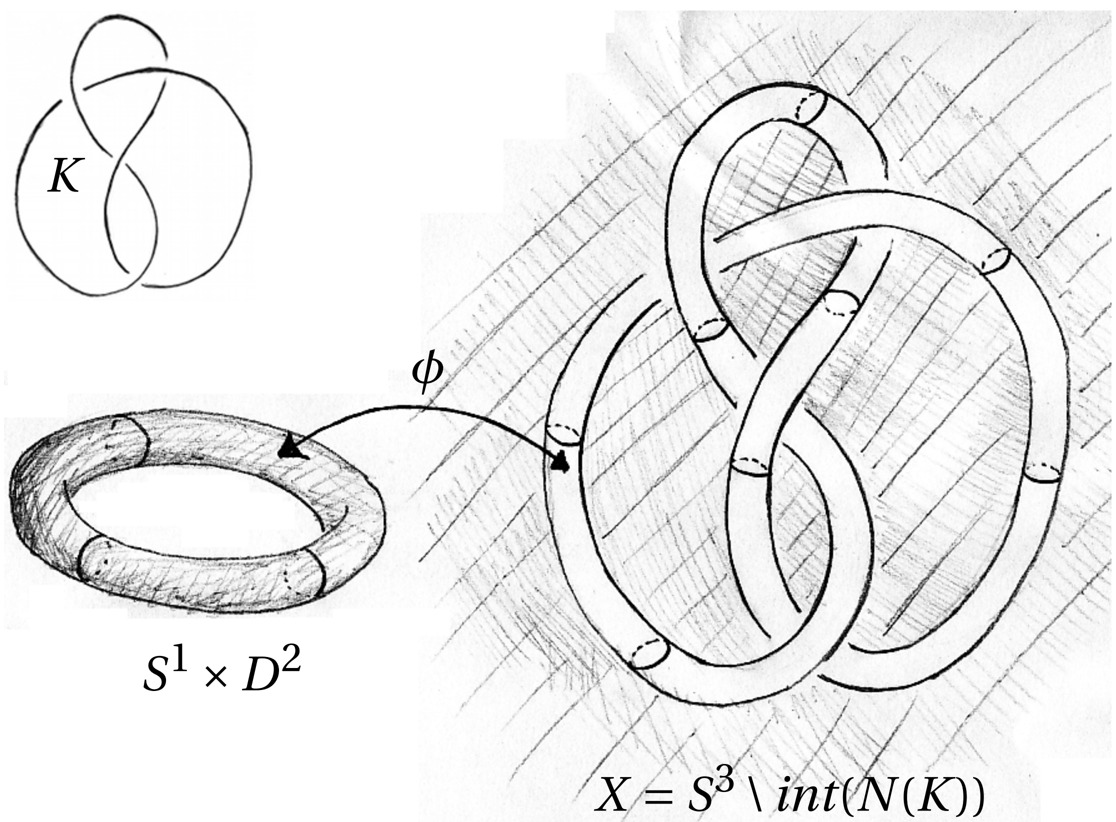

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Closely related question: Does M contain an *immersed* incompressible surface? Equivalently, does $\pi_1(M)$ contain the fundamental group of some surface?

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A lot of evidence for this conjecture including:

- True for all the manifolds coming from the figure-8 knot. [D-Thurston 2003].
- Weaker results for surgery on any knot, e.g. [Cooper-Long 1997, Cooper-Walsh 2006].
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Conj. *Let M be an irreducible compact 3-manifold. If $\pi_1(M)$ is infinite, then M has a* non-trivial *finite cover.*

Equivalently, $\pi_1(M)$ has a subgroup H with $1 < [\pi_1(M) : H] < \infty$.

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Geometrization (Thurston/Perelman):
A compact M^3 can be cut along spheres and incompressible tori into pieces which admit geometric structures. That is, each piece admits a homogeneous Riemannian metric modeled on one of

$$\mathbb{E}^3,\ S^3,\ \mathbb{H}^3,\ S^2 \times \mathbb{R},\ \mathbb{H}^2 \times \mathbb{R},\ \mathrm{Nil},\ \mathrm{Sol},\ \widetilde{\mathrm{SL}_2\mathbb{R}}.$$

Ex: T^3 is Euclidean as $= \mathbb{E}^3/\mathbb{Z}^3$, whereas $S^2 \times S^1$ has a $S^2 \times \mathbb{R}$ geometry.

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From now on, M will be a *hyperbolic* 3-manifold, i.e. one with a metric of constant sectional curvature -1. Equivalently, $M=\mathbb{H}^3/\Gamma$, where $\Gamma\leq\mathrm{Isom}^+(\mathbb{H}^3)=\mathrm{M\ddot{o}bius}(\hat{\mathbb{C}})=\mathrm{PSL}_2(\mathbb{C})$.

Here $\mathbb{H}^3=\left\{\mathbf{x}\in\mathbb{R}^3 \,\middle|\, |\mathbf{x}|<1\right\}$ with the metric where

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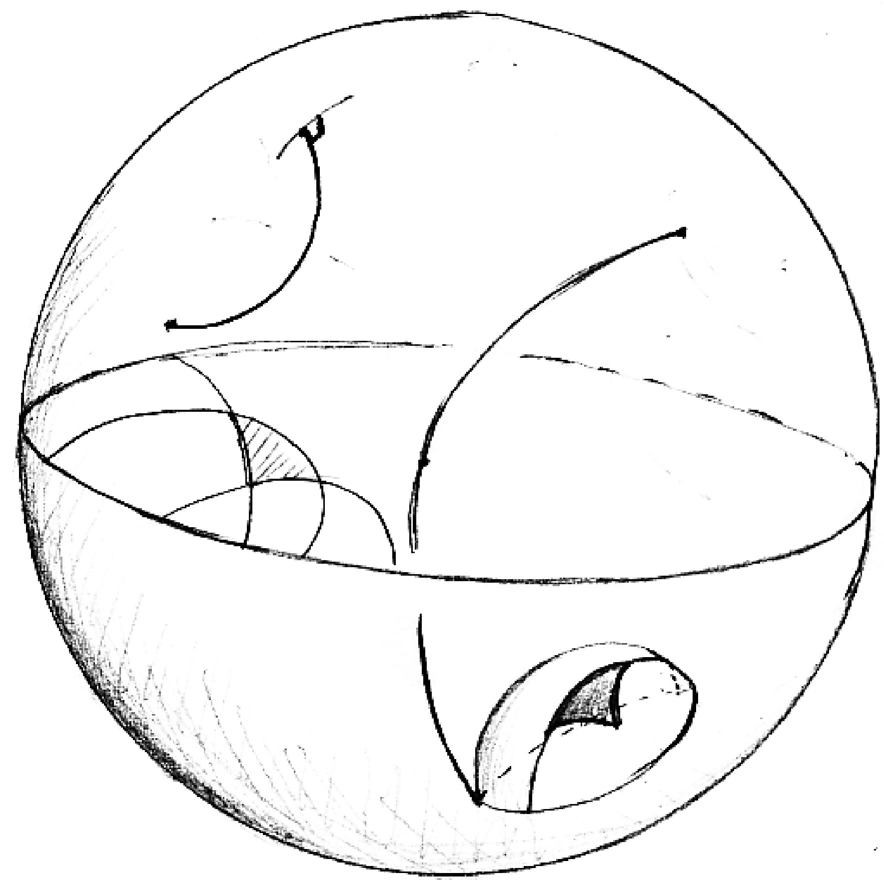

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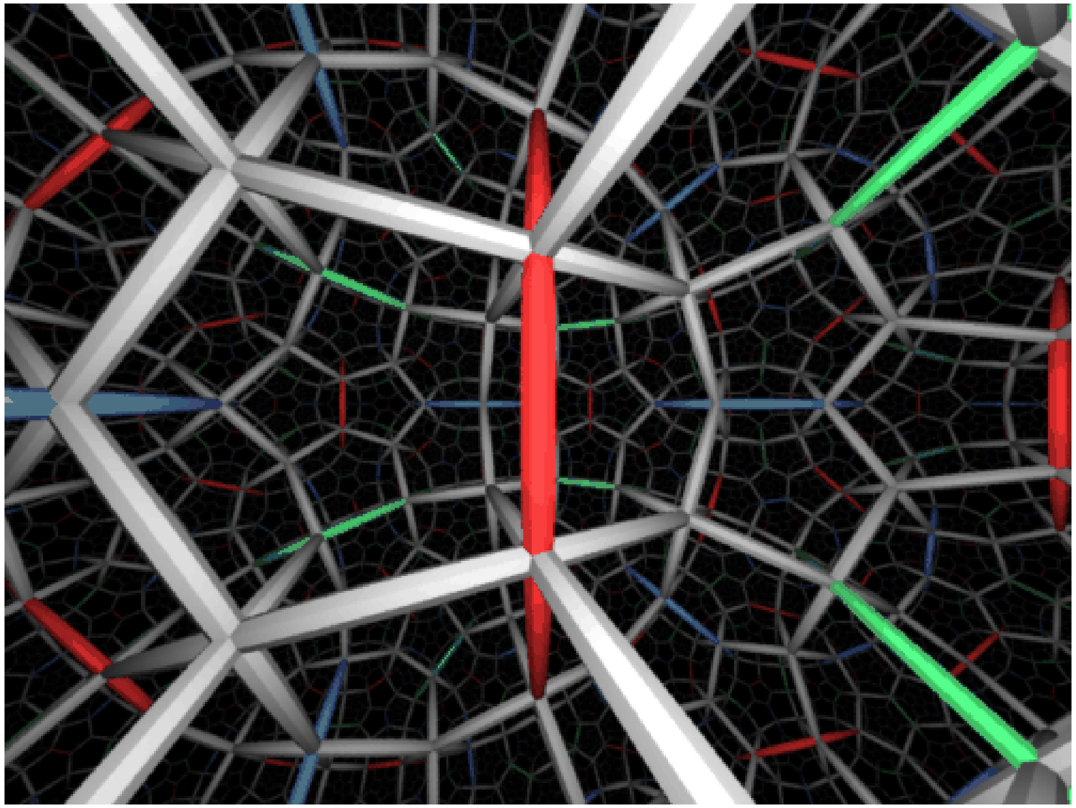

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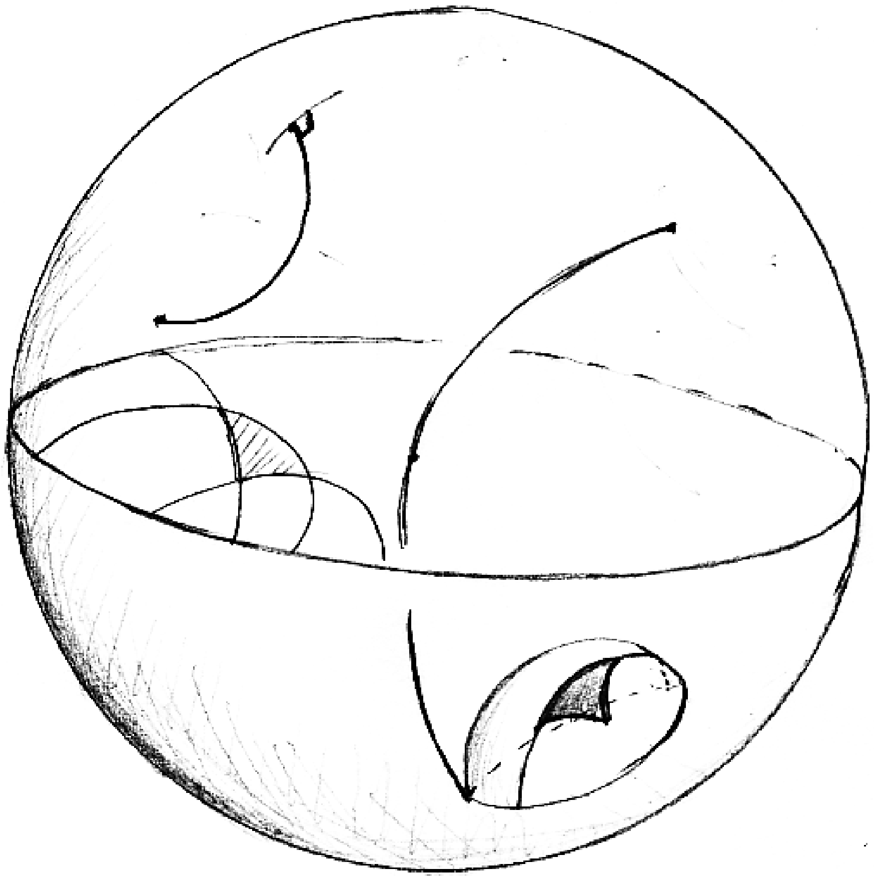

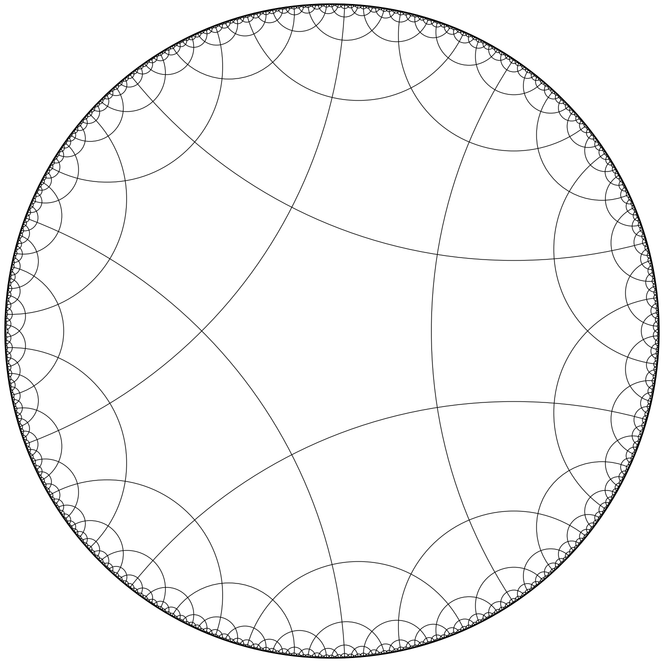

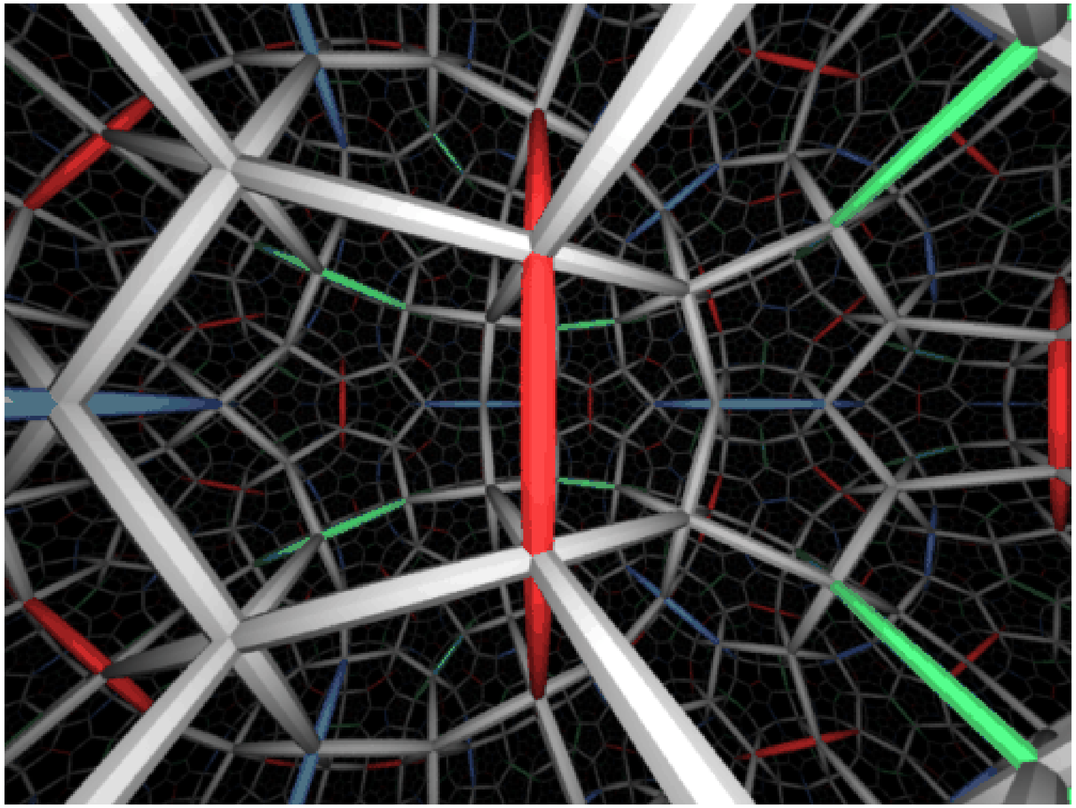

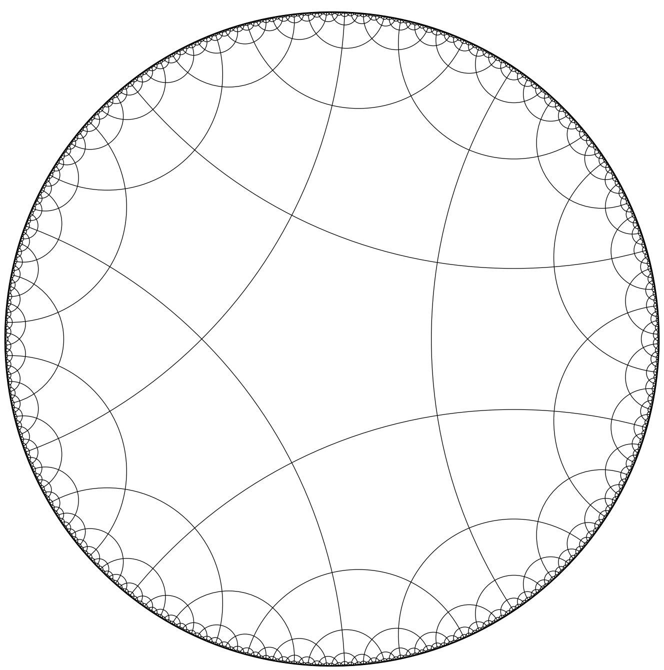

Thm (Perelman 2003). *Let M be a compact 3-manifold. If $\pi_1(M)$ is infinite, then M has a non-trivial finite cover. Equivalently, $\pi_1(M)$ has a finite-index proper subgroup.*

Proof. Reduce to the case when M is hyperbolic. As M is compact, $\pi_1(M)$ is finitely generated and also $\pi_1(M) \leq \mathrm{PSL}_2(\mathbb{C})$. A finitely generated group of matrices has many finite index subgroups by [Mal'tsev 1940s]. Idea: For $\mathrm{PSL}_2(\mathbb{Z})$ we build the needed subgroup Λ by considering:

$$1 \to \Lambda \to \mathrm{PSL}_2(\mathbb{Z}) \to \mathrm{PSL}_2(\mathbb{Z}/(p\mathbb{Z})) \to 1.$$

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[Lubotzky 1995] Many more subgroups than just the congruence ones.

[D-Thurston 2006] Studied random Heegaard splittings. For a finite simple group Q, the number of Q-covers is Poisson distributed with mean

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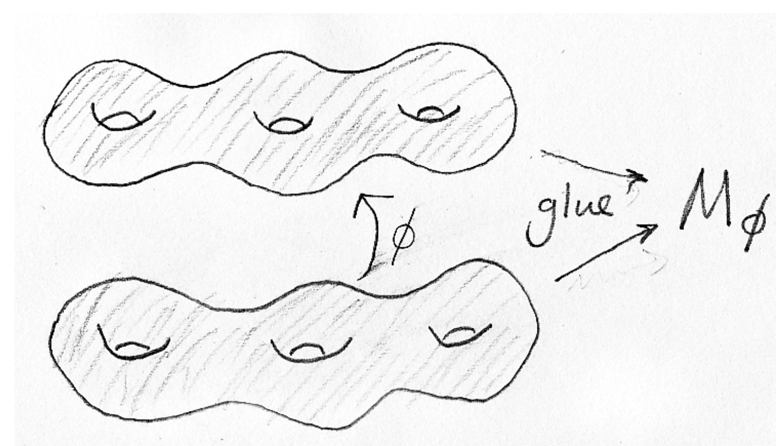
E.g. the probability of an A_n cover is $1 - e \approx 0.6$.



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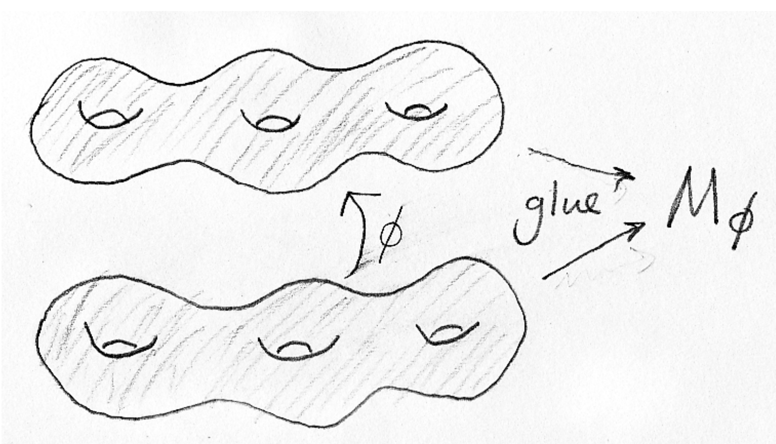

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$$M \leftarrow M_1 \leftarrow M_2 \leftarrow M_3 \leftarrow \cdots$$

exhausts M if $\bigcap \pi_1(M_n) = 1$.

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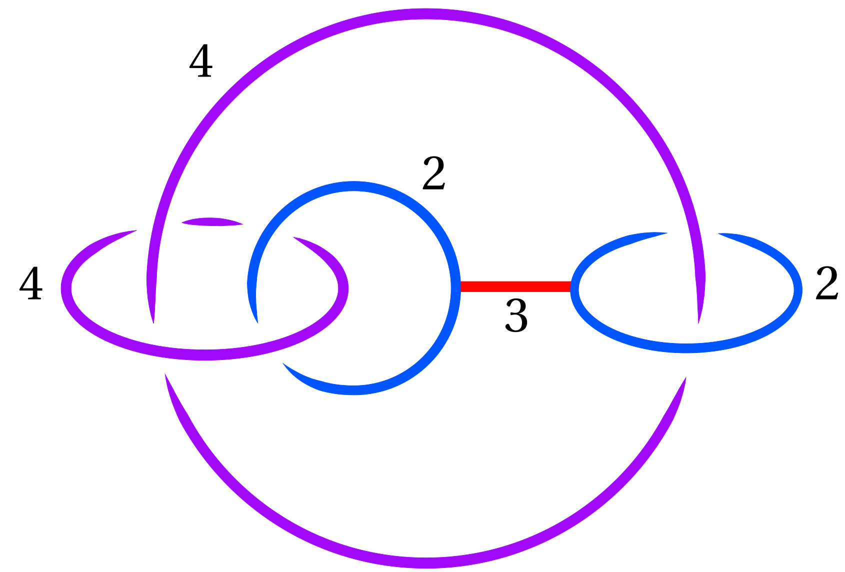
Proof conditional on Langlands for GL_2 and the Generalized Riemann Hypothesis!

Thankfully, Boston-Ellenberg (2006) were able to analyze these examples unconditionally, using our picture:



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