Hyperbolically twisted Alexander polynomials of knots

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Gauge Theory Seminar, April 6, 2012

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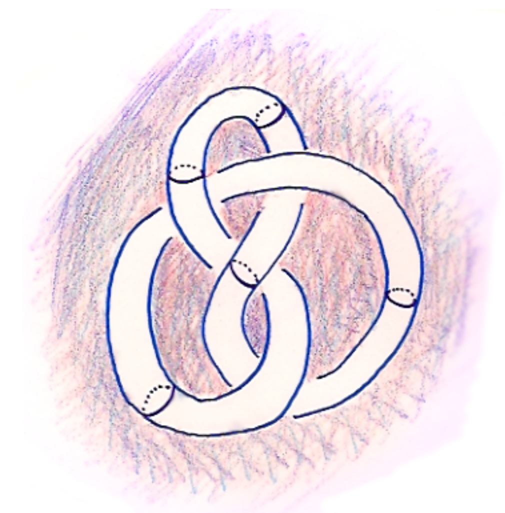
Setup:

- Knot: $K = S^1 \hookrightarrow S^3$
- Exterior: $M = S^3 - \mathring{N}(K)$



A basic and fundamental invariant of K its *Alexander polynomial* (1923):

$$\Delta_K(t) = \Delta_M(t) \in \mathbb{Z}[t, t^{-1}]$$

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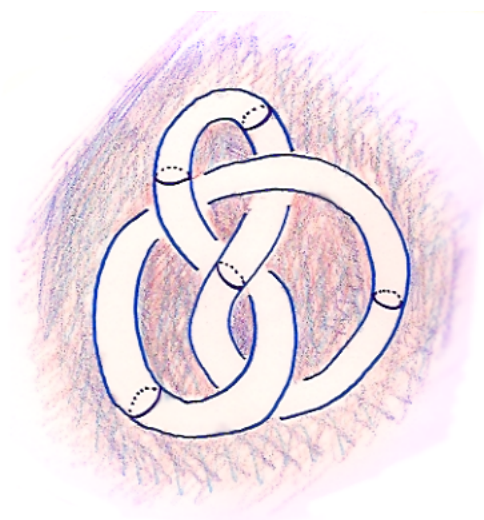

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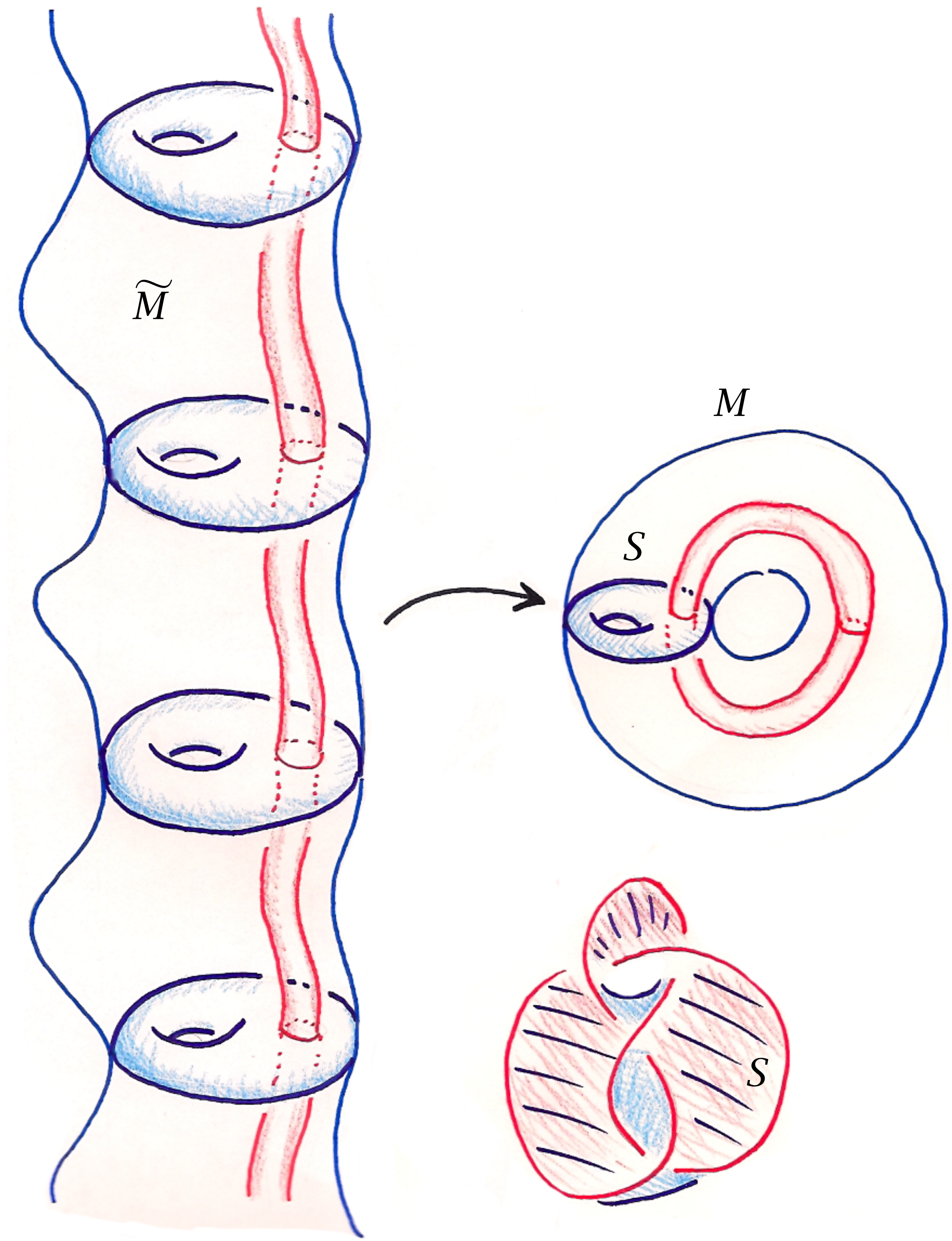
Universal cyclic cover: corresponds to the kernel of the unique epimorphism $\pi_1(M) \to \mathbb{Z}$.



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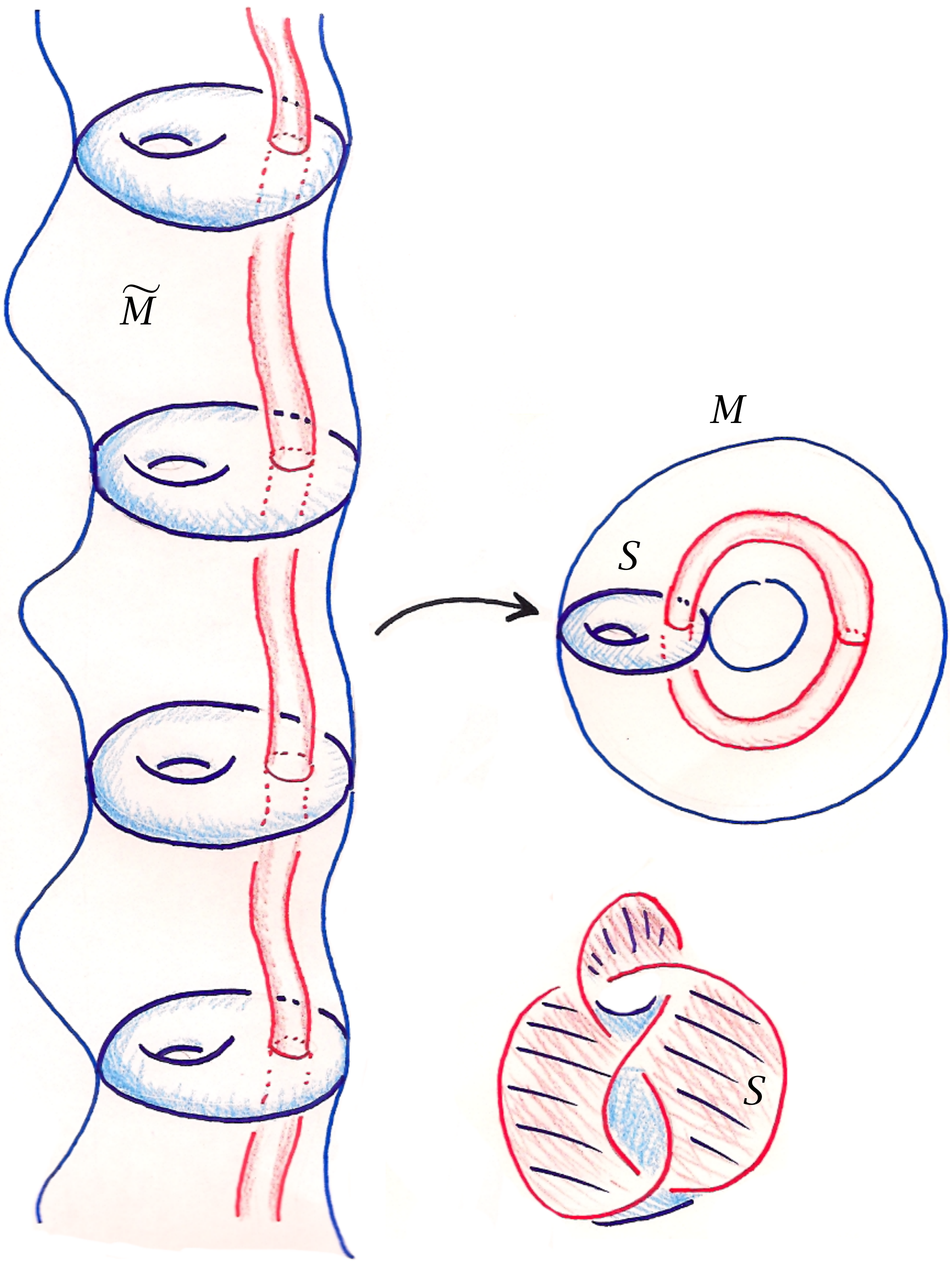

$A_M = H_1(\widetilde{M};\mathbb{Q})$ is a module over $\Lambda = \mathbb{Q}[t^{\pm 1}]$, where $\langle t \rangle$ is the covering group.

As Λ is a PID,

$$A_M = \prod_{k=0}^{n} \Lambda \big/ (p_k(t))$$

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Figure-8 knot:

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Genus:

$$\begin{aligned} g &= \min\left(\text{genus of } S \text{ with } \partial S = K\right) \\ &= \min\left(\text{genus of } S \text{ gen. } H_2(M, \partial M; \mathbb{Z})\right) \end{aligned}$$

Fundamental fact:

$$2g \geq \deg(\Delta_M)$$

Proof: Note $\deg(\Delta_M) = \dim_{\mathbb{Q}}(A_M)$. As A_M is generated by $H_1(S;\mathbb{Q}) \cong \mathbb{Q}^{2g}$, the inequality follows.

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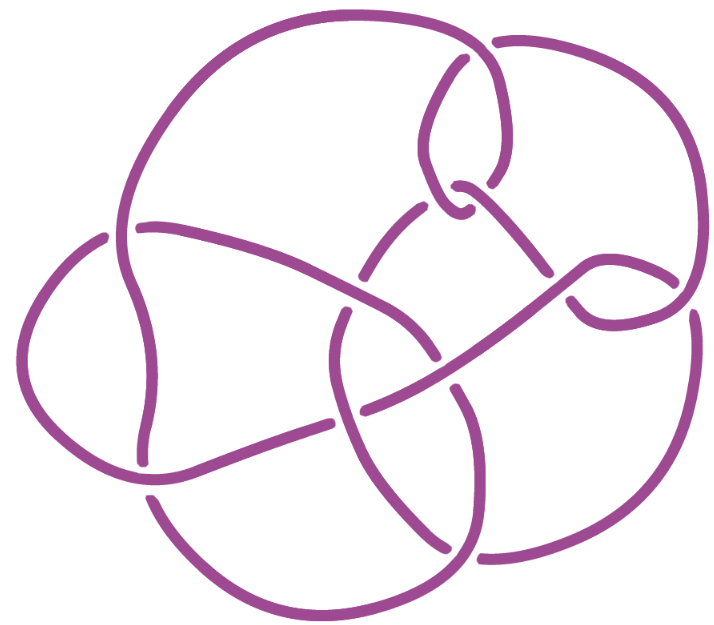
$\Delta(t)$ determines g for all alternating knots and all fibered knots.

Kinoshita-Terasaka knot: $\Delta(t) = 1$ but $g = 2$.



Idea: Improve Δ_M by looking at $H_1(\widetilde{M}; V_\rho)$ for the system of local coefficients coming from a representation $\alpha\colon \pi_1(M) \to \mathrm{GL}(V)$. [Lin 1990; Wada 1994,...]

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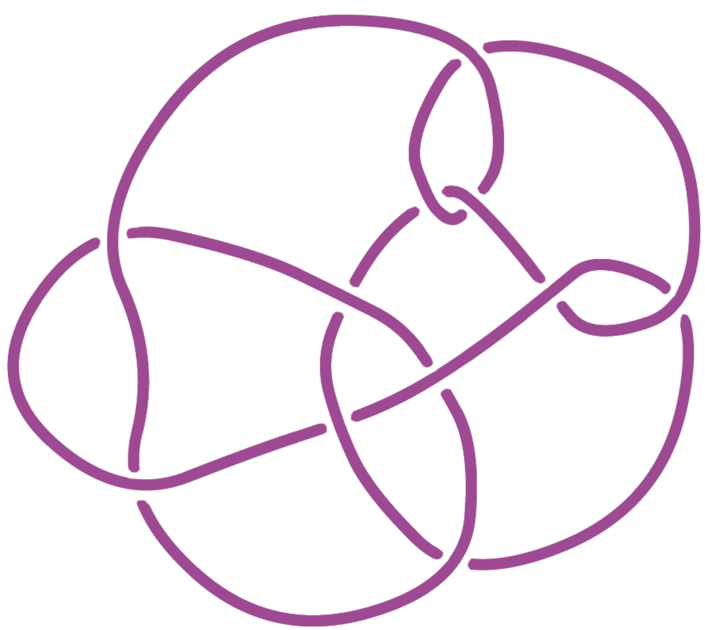

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Technically, it's best to define $\tau_{M,\alpha}$ as a torsion, a la Reidemeister/Milnor/Turaev.

Genus bound: When α is irreducible and non-trivial:

$$2g - 1 \geq \frac{1}{\dim V} \deg(\tau_{M,\alpha}) \qquad (\star)$$

Proof:

$$\begin{aligned} \deg(\tau_{M,\alpha}) &= \dim H_1(\widetilde{M}; V_\alpha) \\ &\leq \dim H_1(S; V_\alpha) = (\dim V) \cdot \lvert \chi(S) \rvert \end{aligned}$$

Thm (Friedl-Vidussi, using Agol and Wise)

If M is hyperbolic, then there exists some α where $(\star)$ is sharp.

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Assumption: M is hyperbolic, i.e.

$$\mathring{M} = \mathbb{H}^3 / \Gamma \quad \text{for a lattice } \Gamma \leq \mathrm{Isom}^+ \mathbb{H}^3$$

Thus have a faithful representation

$$\alpha\colon \pi_1(M) \to \mathrm{SL}_2\mathbb{C} \leq \mathrm{GL}(V) \quad \text{where } V = \mathbb{C}^2.$$

Hyperbolic Alexander polynomial:

$$\tau_M(t) \in \mathbb{C}[t^{\pm 1}] \quad \text{coming from } H_1(\widetilde{M}; V_\alpha).$$

Examples:

- Figure-8: $\tau_M = t - 4 + t^{-1}$
- Kinoshita-Terasaka:

$$\begin{aligned} \tau_M \approx &(4.417926 + 0.376029i)(t^3 + t^{-3}) \\ &- (22.941644 + 4.845091i)(t^2 + t^{-2}) \\ &+ (61.964430 + 24.097441i)(t + t^{-1}) \\ &- (-82.695420 + 43.485388i) \end{aligned}$$

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Basic Properties:

- τ_M is an unambiguous element of $\mathbb{C}[t^{\pm 1}]$ with $\tau_M(t) = \tau_M(t^{-1})$.
- The coefficients of τ_M lie in $\mathbb{Q}(\mathrm{tr}(\Gamma))$ and are often algebraic integers.
- $\tau_M(\zeta) \neq 0$ for any root of unity ζ.
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- M amphichiral $\Rightarrow \tau_M(t) \in \mathbb{R}[t^{\pm 1}]$.
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313,231	number of prime knots with at most 15 crossings. [HTW 98]
22	number which are non-hyperbolic.
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Conj. *τ_M determines the genus and fibering for any hyperbolic knot in S^3.*

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Approach 1: Deform the representation

Can consider other reps to $SL_2\mathbb{C}$, understand how $\tau_{M,\alpha}$ varies as you move around the character variety:

Example: $m037$, $X_0 = \mathbb{C} \setminus \{-2, 0, 2\}$

$$\tau_{X_0}(t) = \frac{(u+2)^4}{16u^2}\left(t + t^{-1}\right) + \frac{(u+2)(u^4+4u^3-8u^2+16u+16)}{8\,(u-2)u^2}$$

Can sometimes connect this universal polynomial to Δ_M.

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Approach 2: Use adjoint representation

$\mathrm{Isom}^+(\mathbb{H}^3) = \mathrm{PSL}_2(\mathbb{C}) \to \mathrm{Aut}(\mathfrak{sl}_2) \leq \mathrm{SL}_3\mathbb{C}$ to get τ_M^{adj} (Dubois-Yamaguchi).

Point: $T_{[\alpha]}X(\pi_1(M)) = H^1(M, (\mathfrak{sl}_2)_{\mathrm{adj}\circ\alpha})$

8,834	knots where $2g > \deg(\Delta_M)$.
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12	knots where $6g - 9 \geq \deg\left(\tau_M^{adj}\right)$.
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Geometric isolation phenomena

Approach 3: Gauge theory

[Kronheimer-Mrowka] Instanton Floer homology detects the genus!