# Hyperbolically twisted Alexander polynomials of knots


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This talk available at http://dunfield.info/

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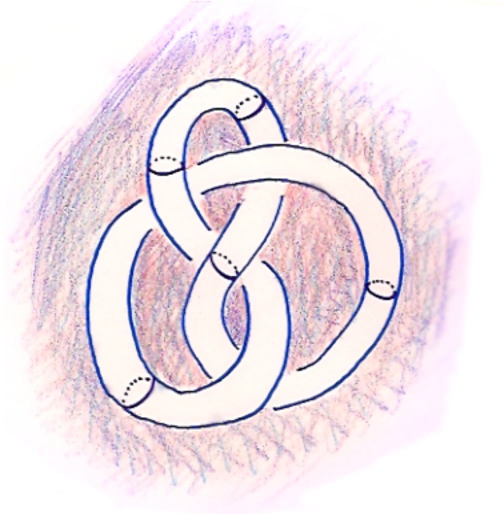
Setup:

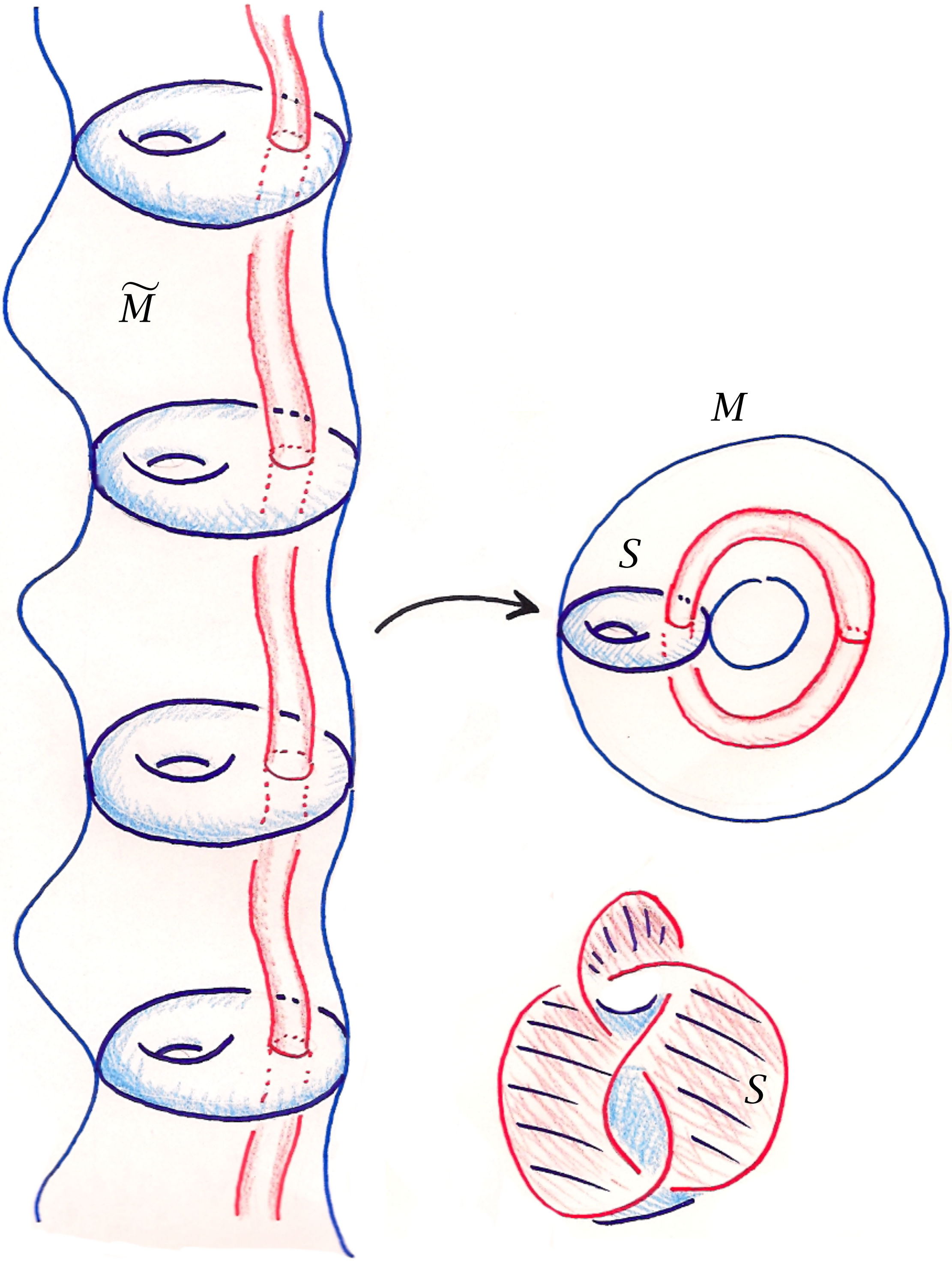
- Knot: $K = S^1 \hookrightarrow S^3$
- Exterior: $M = S^3 - \mathring{N}(K)$



A basic and fundamental invariant of $K$ its *Alexander polynomial* (1923):

$$\Delta_K(t) = \Delta_M(t) \in \mathbb{Z}[t, t^{-1}]$$

Universal cyclic cover: corresponds to the kernel of the unique epimorphism $\pi_1(M) \to \mathbb{Z}$.



$A_M = H_1(\widetilde{M};\mathbb{Q})$ is a module over $\Lambda = \mathbb{Q}[t^{\pm 1}]$, where $\langle t \rangle$ is the covering group.

As $\Lambda$ is a PID,

$$A_M = \prod_{k=0}^{n} \Lambda \Big/ (p_k(t))$$

Define

$$\Delta_M(t) = \prod_{k=0}^{n} p_k(t) \in \mathbb{Q}[t, t^{-1}]$$

Figure-8 knot:

$$\Delta_M = t - 3 + t^{-1}$$

Genus:

$$\begin{aligned} g &= \min\big(\text{genus of } S \text{ with } \partial S = K\big) \\ &= \min\big(\text{genus of } S \text{ gen. } H_2(M, \partial M; \mathbb{Z})\big) \end{aligned}$$

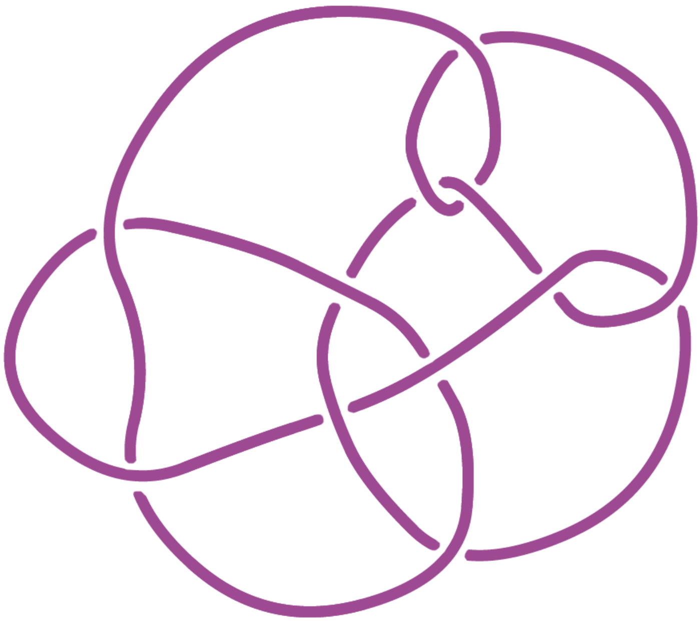
Fundamental fact:

$$2g \geq \deg(\Delta_M)$$

Proof: Note $\deg(\Delta_M) = \dim_{\mathbb{Q}}(A_M)$. As $A_M$ is generated by $H_1(S;\mathbb{Q}) \cong \mathbb{Q}^{2g}$, the inequality follows.

$\Delta(t)$ determines $g$ for all alternating knots and all fibered knots.

Kinoshita-Terasaka knot: $\Delta(t) = 1$ but $g = 2$.



Idea: Improve $\Delta_M$ by looking at $H_1(\widetilde{M}; V_\rho)$ for the system of local coefficients coming from a representation $\alpha\colon \pi_1(M) \to \mathrm{GL}(V)$. [Lin 1990; Wada 1994,...]

Twisted Alexander polynomial: $\tau_{M,\alpha} \in \mathbb{F}[t^{\pm 1}]$

Technically, it's best to define $\tau_{M,\alpha}$ as a torsion, a la Reidemeister/Milnor/Turaev.

Genus bound: When $\alpha$ is irreducible and non-trivial:

$$2g - 1 \geq \frac{1}{\dim V} \deg(\tau_{M,\alpha}) \tag{$\star$}$$

Proof:

$$\begin{aligned} \deg(\tau_{M,\alpha}) &= \dim H_1(\widetilde{M}; V_\alpha) \\ &\leq \dim H_1(S; V_\alpha) = (\dim V) \cdot |\chi(S)| \end{aligned}$$

**Thm (Friedl-Vidussi, using Agol and Wise)**
*If $M$ is hyperbolic, then there exists some $\alpha$ where $(\star)$ is sharp.*

Idea: By Wise, $\pi_1(M)$ is virtually special, hence RFRS. By Agol, there exists a finite cover of $M$ where the lift of $S$ is a limit of fiberations. Use $\alpha$ associated to this cover.

Assumption: $M$ is hyperbolic, i.e.

$$\mathring{M} = \mathbb{H}^3 \big/ \Gamma \quad \text{for a lattice } \Gamma \leq \mathrm{Isom}^+ \mathbb{H}^3$$

Thus have a faithful representation

$$\alpha\colon \pi_1(M) \to \mathrm{SL}_2\mathbb{C} \leq \mathrm{GL}(V) \quad \text{where } V = \mathbb{C}^2.$$

Hyperbolic Alexander polynomial:

$$\tau_M(t) \in \mathbb{C}[t^{\pm 1}] \quad \text{coming from } H_1(\widetilde{M}; V_\alpha).$$

Examples:

- Figure-8: $\tau_M = t - 4 + t^{-1}$
- Kinoshita-Terasaka:

$$\begin{aligned}\tau_M \approx &(4.417926 + 0.376029i)(t^3 + t^{-3}) \\ &- (22.941644 + 4.845091i)(t^2 + t^{-2}) \\ &+ (61.964430 + 24.097441i)(t + t^{-1}) \\ &- (-82.695420 + 43.485388i)\end{aligned}$$

Basic Properties:

- $\tau_M$ is an unambiguous element of $\mathbb{C}[t^{\pm 1}]$ with $\tau_M(t) = \tau_M(t^{-1})$.
- The coefficients of $\tau_M$ lie in $\mathbb{Q}\big(\operatorname{tr}(\Gamma)\big)$ and are often algebraic integers.
- $\tau_M(\zeta) \neq 0$ for any root of unity $\zeta$.
- $\tau_{\overline{M}} = \overline{\tau_M(t)}$
- $M$ amphichiral $\Rightarrow \tau_M(t) \in \mathbb{R}[t^{\pm 1}]$.
- Genus bound:

$$4g - 2 \geq \deg \tau_M(t)$$

  For the KT knot, $g = 2$ and $\deg \tau_M(t) = 6$ so this is sharp, unlike with $\Delta_M$.

Knots by the numbers:

| | |
|---|---|
| 313,231 | number of prime knots with at most 15 crossings. [HTW 98] |
| 22 | number which are non-hyperbolic. |
| 8,834 | number where $2g > \deg(\Delta_M)$. |
| 7,972 | number of non-fibered knots where $\Delta_M$ is monic. |

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| 0 | number where $4g - 2 > \deg(\tau_M)$. |
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**Conj.** *$\tau_M$ determines the genus and fibering for any hyperbolic knot in $S^3$.*

Computing $\tau_M$: Approximate $\pi_1(M) \to \mathrm{SL}_2\mathbb{C}$ to 250 digits by solving the gluing equations associated to some ideal triangulation of $M$ to high precision.

Genus and fibering for most of these knots was previously unknown; Haken-style normal surface algorithms are impractical in this range, various tricks were used.

**Q. How can we prove this conjecture?**

Not known to be true for infinitely many non-fibered knots!

If conjecture and GRH are true, then knot genus is in **NP** $\cap$ **co-NP**.

**Approach 1:** Deform the representation

Can consider other reps to $SL_2\mathbb{C}$, understand how $\tau_{M,\alpha}$ varies as you move around the character variety:

Example: $m037$, $X_0 = \mathbb{C} \setminus \{-2, 0, 2\}$

$$\tau_{X_0}(t) = \frac{(u+2)^4}{16u^2}\left(t + t^{-1}\right) + \frac{(u+2)(u^4 + 4u^3 - 8u^2 + 16u + 16)}{8\,(u-2)u^2}$$

Can sometimes connect this universal polynomial to $\Delta_M$.

Ideal points corresponding to $S$: not helpful.

**Approach 2:** Use adjoint representation

$\mathrm{Isom}^+(\mathbb{H}^3) = \mathrm{PSL}_2(\mathbb{C}) \to \mathrm{Aut}(\mathfrak{sl}_2) \leq \mathrm{SL}_3\mathbb{C}$

to get $\tau_M^{adj}$ (Dubois-Yamaguchi).

Point: $T_{[\alpha]}X\big(\pi_1(M)\big) = H^1\big(M, (\mathfrak{sl}_2)_{\mathrm{adj}\circ\alpha}\big)$

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| 8,834 | knots where $2g > \deg(\Delta_M)$. |
| | knots where $6g - 3 > \deg\left(\tau_M^{adj}\right)$. |
| 7,972 | non-fibered with $\Delta_M$ monic. |
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| 8,252 | knots where $6g - 3 > \deg\left(\tau_M^{adj}\right)$. |

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| 8,834 | knots where $2g > \deg(\Delta_M)$. |
| 8,252 | knots where $6g - 3 > \deg\left(\tau_M^{adj}\right)$. |
| 12 | knots where $6g - 9 \geq \deg\left(\tau_M^{adj}\right)$. |
| 7,972 | non-fibered with $\Delta_M$ monic. |
| 0 | non-fibered with $\tau_M^{adj}$ monic. |

Geometric isolation phenomena

**Approach 3:** Gauge theory

[Kronheimer-Mrowka] Instanton Floer homology detects the genus!