Annoying trailers:

SnapPy http://snappy.computop.org

What is SnapPy?

SnapPy is a user interface to the SnapPea kernel which runs on Mac OS X, Linux, and Windows. SnapPy combines a link editor and 3D-graphics for Dirichlet domains and cusp neighborhoods with a powerful command-line interface based on the Python programming language. You can see it in action, learn how to install it, and read the tutorial.

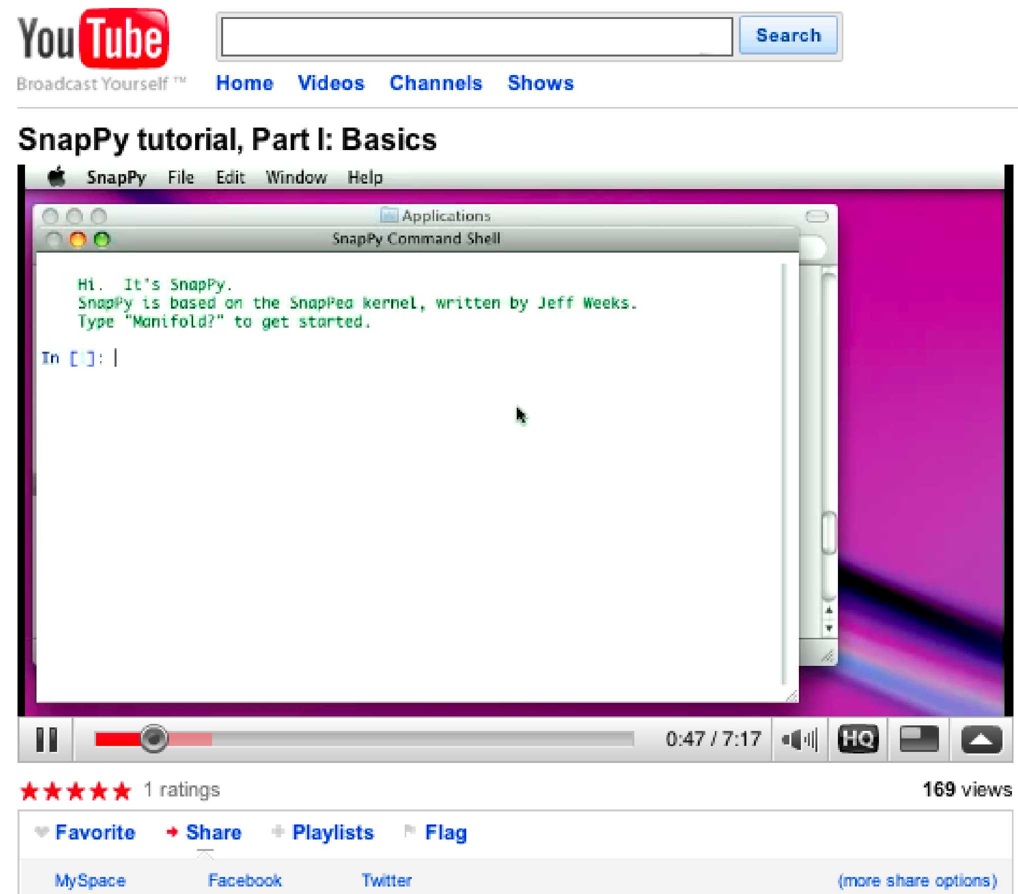
Contents

- Screenshots: SnapPy in action
- Installing and running SnapPy
- Tutorial
- snappy: A Python interface for SnapPea
- Using SnapPy's link editor
- To Do List
- Development Basics: OS X
- Development Basics: Windows XP

Credits

Written by Marc Culler and Nathan Dunfield. Uses the SnapPea kernel written by Jeff Weeks. Released under the terms of the GNU General Public License.

http://www.youtube.com/user/NathanDunfield/



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Nathan M. Dunfield
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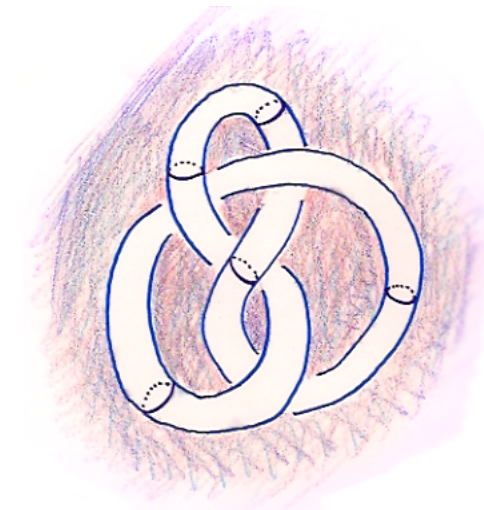
- Exterior: $M = S^3 - \mathring{N}(K)$



A basic and fundamental invariant of K its *Alexander polynomial* (1923):

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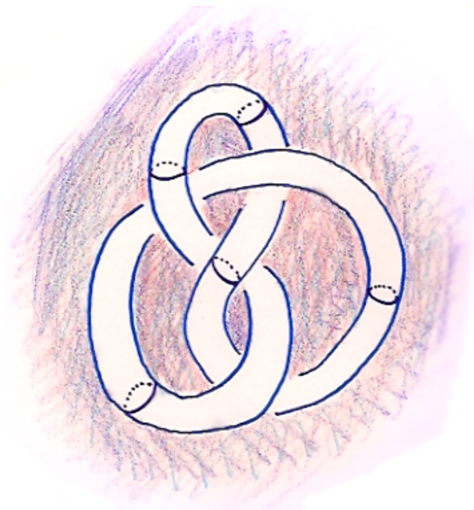

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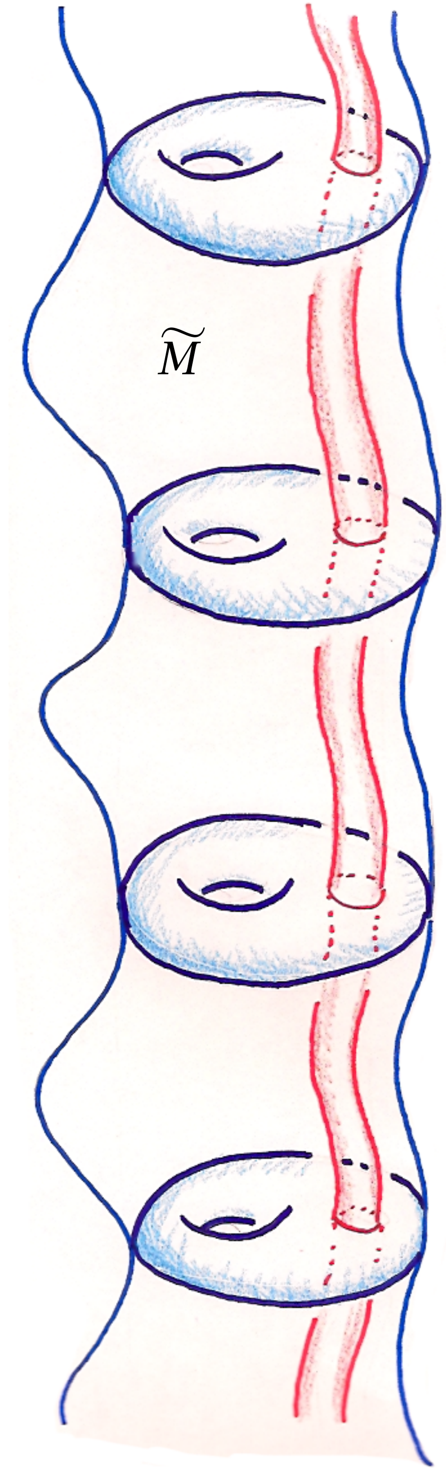

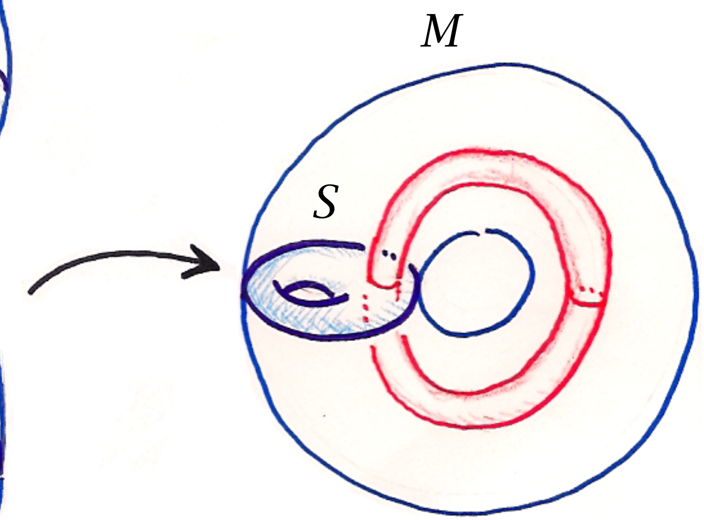

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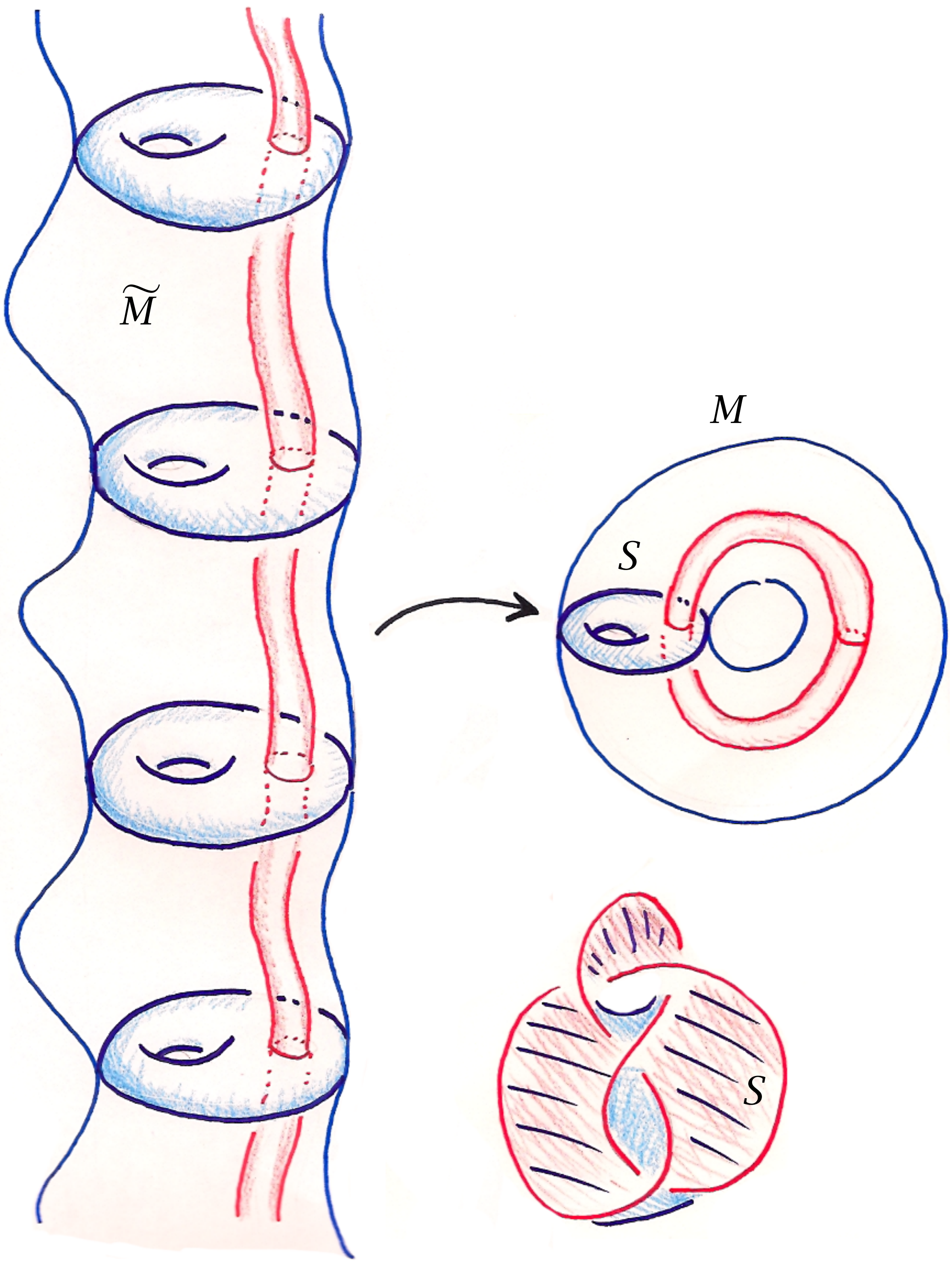

$A_M = H_1(\widetilde{M};\mathbb{Q})$ is a module over $\Lambda = \mathbb{Q}[t^{\pm 1}]$, where $\langle t \rangle$ is the covering group.

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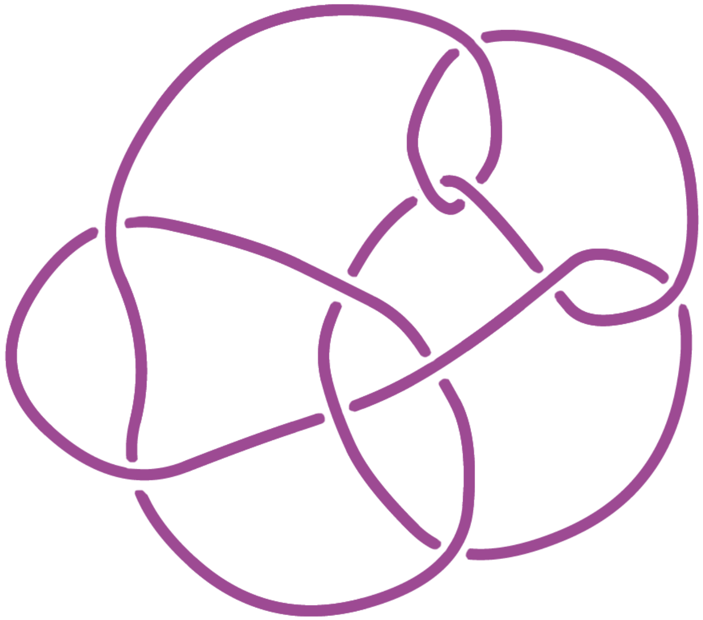

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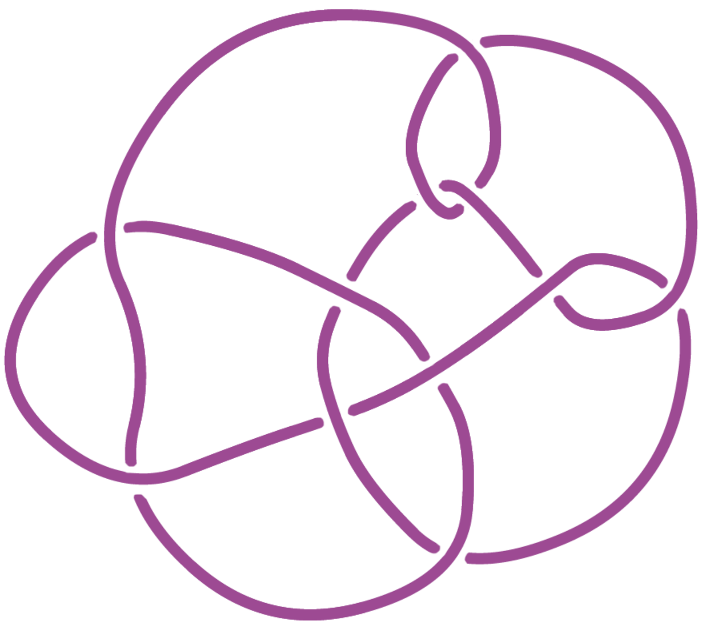

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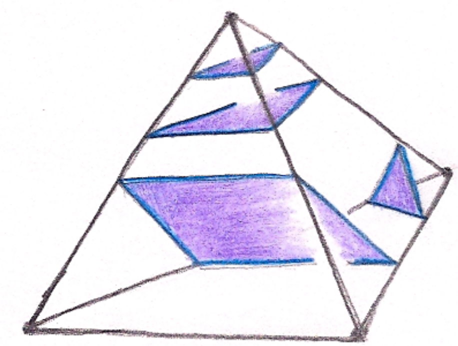

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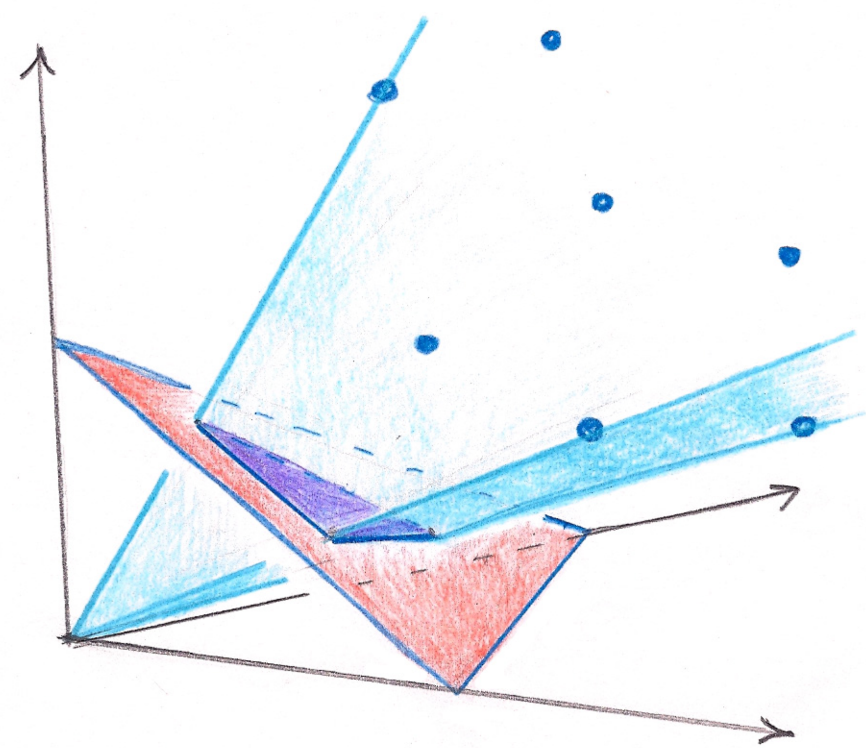

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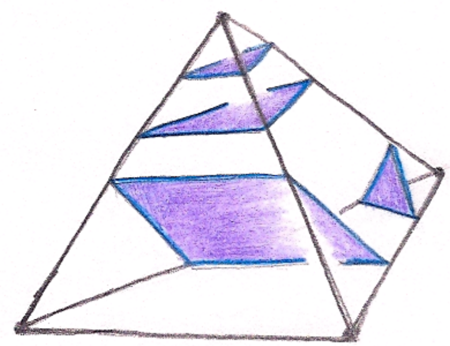

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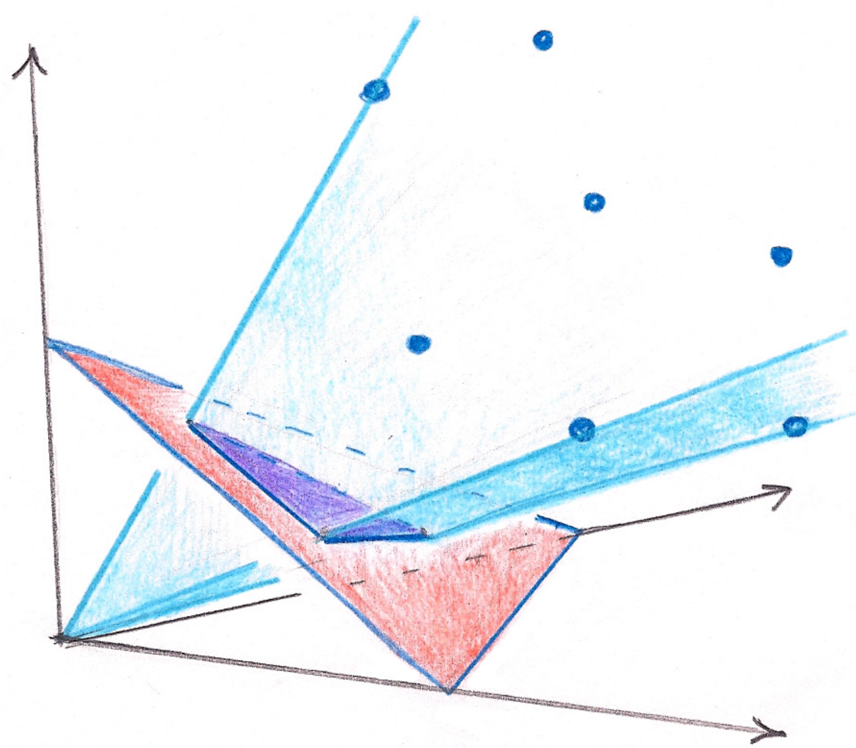

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E.g. The class of minimal genus surfaces whose boundary is a given knot.

Problem: There can be exponentially many vertex rays, typically $\approx O(1.6^t)$ [Burton 2009]. In practice, limited to $t < 40$.

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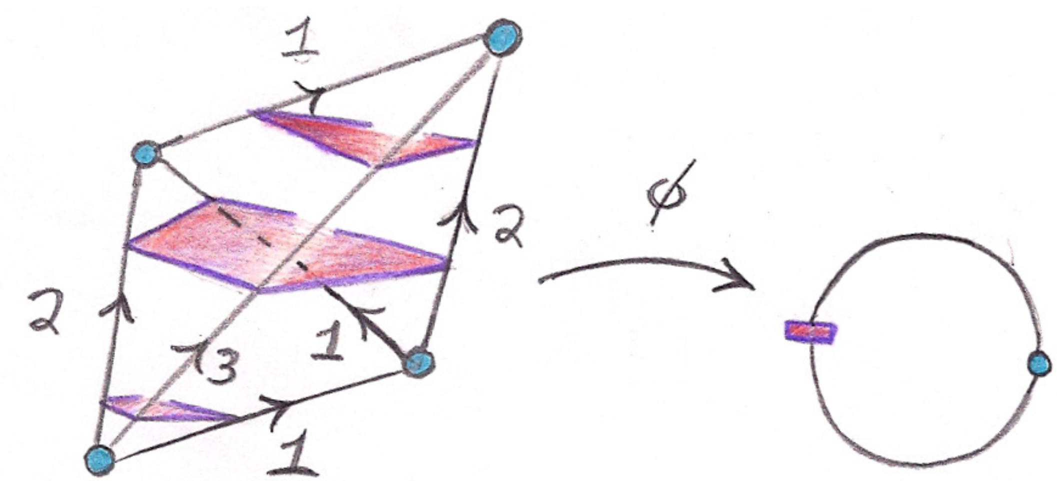

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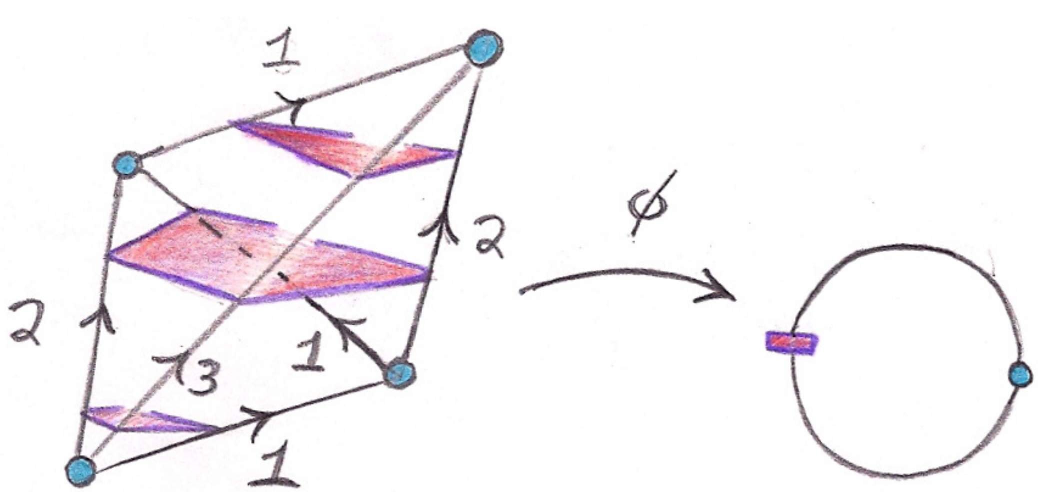

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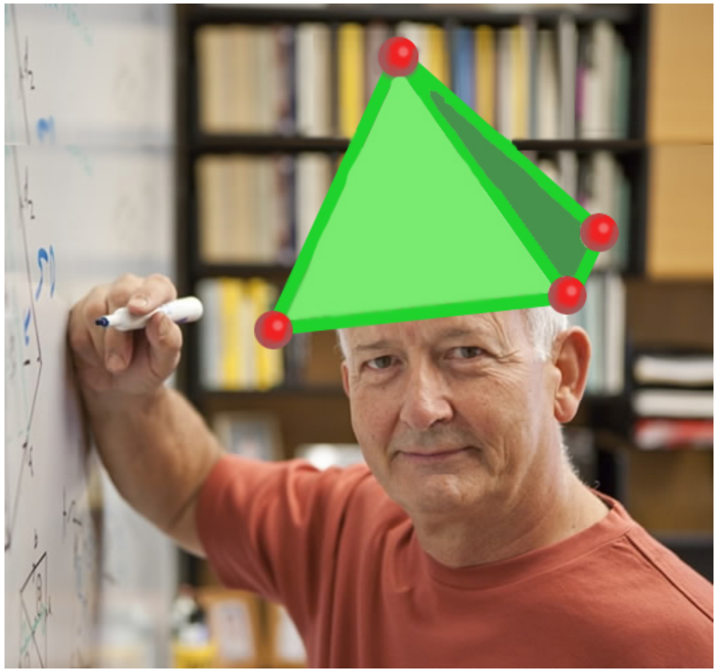

Bus!