Lecture 4:

Previously on CGS...

[Thurston] Suppose $\mathcal{T}$ is an ideal triangulation of M^3. If $z_i \in \mathbb{C}$ with $\mathrm{Im}(z_i) > 0$ satisfy the edge and cusp eqns, then they define a complete hyperbolic structure on M.

Today: How can we prove such z_i exist? (and that we know what they are.)

Want to understand

$$V(\mathcal{T}) = \left\{ z \in \mathbb{C}^n \;\middle|\; \begin{array}{l} z \text{ sat the polynomial edge} \\ \text{and cusp eqns for } \mathcal{T} \end{array} \right\}$$

which is an affine algebraic var defined over $\mathbb{Q}$.

Suppose $z_{hyp} \in V(\mathcal{T})$ sat Thurston's thm. By Mostow, only one hyp str on M, can use to show z_{hyp} is unique. From theorem, z_{hyp} is an isolated pt of $V(\mathcal{T})$ (local rigidity a la Calabi-Weil).

Cor: There is a number field $K \subseteq \mathbb{C}$ so that $z_{hyp} \in K^n$

Pf: Decompose $V(\mathcal{T})$ into irreducible components over $\mathbb{Q}$. The pt z_{hyp} is in some 0-dim'l comp V_0.

Eliminating vars to project V_0 onto the i^{th} coor. gives a 0-dim'l variety in $\mathbb{C}$ <u>defined over $\mathbb{Q}$</u>, and so $p_i(V_0) \subseteq \left(\text{roots of } f_i \in \mathbb{Q}[x]\right)$.

<u>Ex:</u> $S^3 \setminus$  $K = \mathbb{Q}(\alpha)$ $\quad \alpha = \sqrt{3}\, i$

$\alpha^2 + 3 = 0$.

$S^3 \setminus$  $K = \mathbb{Q}(\beta)$ $\quad \beta^3 - \beta + 1$

$\beta \approx 0.6623 + 0.5622 i$

K is called the shape field (= trace field)

<u>Approach 1</u>: Use resultants / Gröbner bases / LLL ... to find K and express the shapes in Z_{hyp} as elts of K.

Given K as $\mathbb{Q}[x] / f(x)$ $\leftarrow$ irred poly in $\mathbb{Q}[x]$

and $z \in K^n$ can rigorously determine if $z \in V(J)$ since can do exact arithmetic in K.

[Won't work for even medium sized examples unless you get very lucky.]

Approach 2: [HIKMOT] Interval Analysis.

$$\mathbb{IR} = \{ \underline{x} = [x_0, x_1] \mid x_i \in \mathbb{Q} \}$$

View $\underline{x}$ as an "enclosure" of some unknown $x \in \underline{x}$.

$$\underline{x} + \underline{y} = \{ x + y \mid x \in \underline{x},\ y \in \underline{y} \} = [x_0 + y_0,\ x_1 + y_1]$$

Similarly for other ops: $\times, -, \div$. In practice, round endpoints but never lie: $\underline{x} * \underline{y} \supseteq \{ x * y \mid x \in \underline{x},\ y \in \underline{y} \}$.

An interval extension $\underline{f} : \mathbb{IR} \to \mathbb{IR}$ of $f : \mathbb{R} \to \mathbb{R}$ must satisfy $\underline{f}(\underline{x}) \supseteq \{ f(x) \mid x \in \underline{x} \}$

Ex: $\underline{\sin}([1.1, 1.2]) = [0.892, 0.933]$

Issue: Set $d(\underline{x}) = x_1 - x_0$

Have $d(\underline{x} + \underline{y}) = d(\underline{x}) + d(\underline{y}) \rightsquigarrow$ intervals fuzz out as we do more ops.

Can say $\underline{x} \neq \underline{y}$ when $\underline{x} \cap \underline{y} = \emptyset$ but $\underline{x} = \underline{y}$ is not allowed.

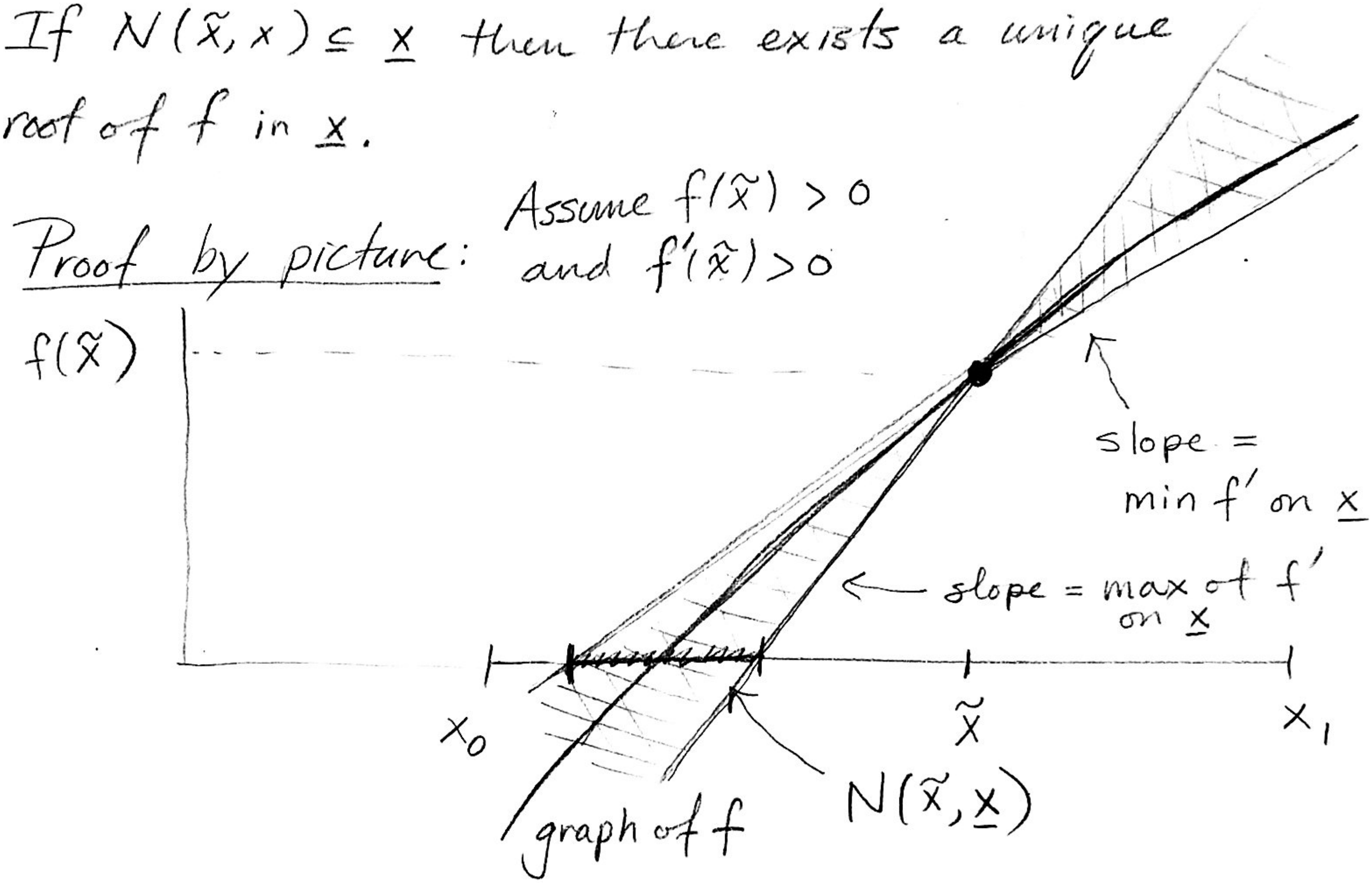
Point: The proof of the Inverse Fn Thm is effective and can be used to show there is a point in $V(J)$ in some small $\underline{z} \in (\mathbb{IC})^n$.

<u>Thm</u>: Suppose $f: \mathbb{R} \to \mathbb{R}$ is differentiable. Given $\underline{x} \in [x_0, x_1] \in \mathbb{IR}$ with $0 \notin \underline{f}'(\underline{x})$ define for $\tilde{x} \in [x_0, x_1]$ the quantity

$$N(\tilde{x}, \underline{x}) = \tilde{x} - \frac{\underline{f}(\tilde{x})}{\underline{f}'(\underline{x})} \in \mathbb{IR}.$$

If $N(\tilde{x}, x) \subseteq \underline{x}$ then there exists a unique root of f in $\underline{x}$.

<u>Proof by picture</u>: Assume $f(\tilde{x}) > 0$ and $f'(\tilde{x}) > 0$



<u>Point</u>: Slope of graph of f on $\underline{x}$ is constrained by $\underline{f}'(\underline{x})$. Thus, f must have a root in $N(\tilde{x}, \underline{x})$.

There exist analogs for $F: \mathbb{R}^n \to \mathbb{R}^n$ or $\mathbb{C}^n \to \mathbb{C}^n$ called "interval Newton's method" and "Krawczyk's test." Rely on Brouwer's Fixed Point Theorem.

[Neumann - Zagier] $V(J)$ can be cut out by # tet equations.

[Demo: SnapPy and SageMath are friends]

The End