Math 526: Algebraic Topology II.

Previously on Math 525:

$\pi_1(X) =$ hom. classes of maps $S^1 \longrightarrow X$

Homology: $H_k(X;G)$

[k dim'l things w/o ∂ / ∂ of k+1 dim'l things]

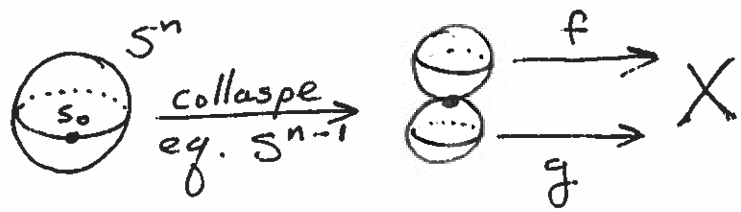
Next on Math 526:

Higher homotopy groups: $\pi_n(X) =$ hom classes of $S^n \longrightarrow X$

Cohomology: $H^k(X;G)$

[Algebraic dual to homology]

Higher homotopy gps: $\pi_n(X, x_0) =$ homotopy classes of $(S^n, s_0) \longrightarrow (X, x_0)$

Group op: $[f] * [g] =$ 

Fact: This is abelian!

n	1	2	3	4	5	6	7
$\pi_n(S^1)$	$\mathbb{Z}$	0	0	0	0	0	0
$\pi_n(S^2)$	0	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/12$	$\mathbb{Z}/2$

Good: Sees most of the homotopy type of X.

Whitehead: Suppose $f: X \to Y$ is a map of CW complexes such that $f_*: \pi_n(X) \longrightarrow \pi_1(Y)$ is $\cong$ for all n. Then X and Y are homotopy equivalent.

<u>Bad:</u> Really hard to compute: Don't know all $\pi_n(S^2)$! Reason is excision fails.

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<u>Cohomology versus homology:</u> Both Top $\longrightarrow$ AbGps

but $X \xrightarrow{f} Y$ gives $H^k(X) \xleftarrow[f^*]{} H^k(Y)$ (contra-variant)

instead of $H_k(X) \xrightarrow[f_*]{} H_k(Y)$ (covariant)

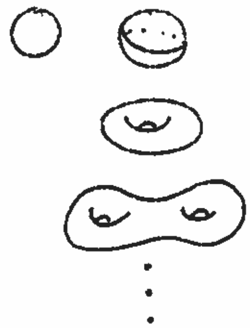
<u>Similarities:</u> H^k is defined by a chain complex,
long exact sequence of pair,
excision, MV sequence ...

Hom and Cohom determine each other; for a field F have $H_k(X;F) \cong H^k(X;F)$.

<u>Key:</u> $H^*(X)$ has a multipication! [Add'l info, also helps with computations.]

$$H^i(X) \times H^j(X) \longrightarrow H^{i+j}(X) \quad \text{[cup product]}$$

Manifolds: Spaces locally homeo to $\mathbb{R}^n$



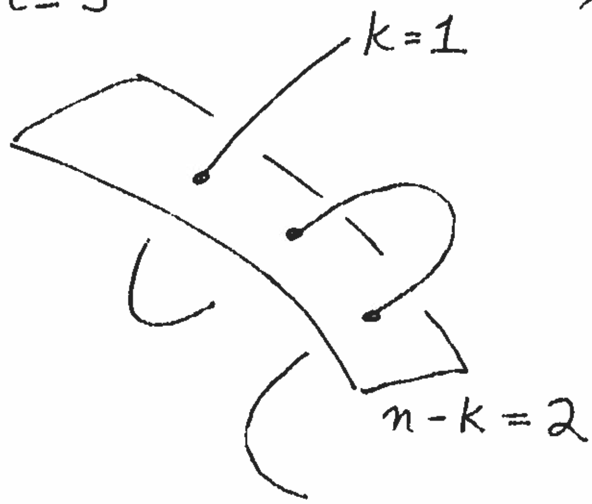
S^3 S^4 ...

On smooth manifolds where one can do calculus, $H^k(M;\mathbb{R})$ can be defined in terms of differential forms, e.g. $\omega = x\, dy \wedge dz - y\, dx \wedge dz + z\, dx \wedge dy$ is a gen for $H^2(S^2;\mathbb{R}) \cong \mathbb{R}$.

Poincaré Duality: Suppose M is a cpt n-manifold. Then $H_k(M;\mathbb{Z}/2) \cong H_{n-k}(M;\mathbb{Z}/2)$. If M is orientable, then $H_k(M;G) \cong H^{n-k}(M;G)$

Source is $H_k(M;\mathbb{Z}/2) \times H_{n-k}(M;\mathbb{Z}/2) \longrightarrow \mathbb{Z}/2$ which counts intersections mod 2. PD $\Leftrightarrow$ this is nondengen.

$n=3$



Dual to cup product

$$H^{n-k} \times H^k \longrightarrow H^n = \mathbb{Z}/2$$

Cohomology 101: X a space

Homology: $\longrightarrow C_{n+1} \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1} \xrightarrow{\partial_{n-1}} \cdots$

$C_n = C_n(X;\mathbb{Z}) = \bigoplus_{\sigma} \mathbb{Z}$ [Singular / Cellular]

$\sigma \leftarrow$ simplex

Cochains: $C^n(X;G) = \mathrm{Hom}(C_n, G) = \prod_{\sigma} G$

$=$ fns from the set of simplices to G

$$\longleftarrow C^{n+1} \xleftarrow{\delta_n} C^n \xleftarrow{\delta_{n-1}} C^{n-1} \longleftarrow$$

Coboundary: For $(\varphi : C_n \to G) \in C^n$ set $\delta_n(\varphi) = \varphi \circ \partial_{n+1}$

Check: $\delta_n \circ \delta_{n-1}(\varphi) = \delta_n(\varphi \circ \partial_n) = \varphi \circ \partial_n \circ \partial_{n+1}$

$= \varphi \circ (\text{zero map}) = 0.$

Def: $H^n(X;G) = \ker \delta_n \big/ \mathrm{im}\, \delta_{n-1}$

Note: $X \xrightarrow{f} Y$

$C_n(X) \xrightarrow{f_*} C_n(Y)$

$C^n(X) \xleftarrow{f^*} C^n(Y)$

Where $f^*(\varphi : C_n(Y) \to G)$
$= \varphi \circ f_*$

Get map on H^n since f^* is a chain map.

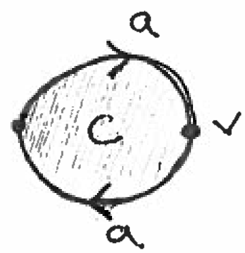
Ex: S^1

$$C_*(S^1;\mathbb{Z}):\quad 0 \longrightarrow \mathbb{Z} \xrightarrow{0} \mathbb{Z} \longrightarrow 0$$

$$C^*(S^1;\mathbb{Z}):\quad 0 \longleftarrow \mathbb{Z} \xleftarrow[0]{} \mathbb{Z} \longleftarrow 0$$

$$H^k(S^1;\mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{for } k=0,1 \\ 0 & \text{otherwise} \end{cases}$$

Ex: $\mathbb{RP}^2$



$$C_*:\quad 0 \longrightarrow \overset{2}{\mathbb{Z}} \xrightarrow[\times 2]{} \overset{1}{\mathbb{Z}} \xrightarrow[0]{} \overset{0}{\mathbb{Z}} \longrightarrow 0$$

$$H_*:\quad 0 \qquad \mathbb{Z}/2 \qquad \mathbb{Z}$$

$$C^*:\quad 0 \longleftarrow \underset{\langle \alpha \rangle}{\underset{\parallel}{\mathbb{Z}}} \xleftarrow{\delta_1 = \times 2} \underset{\langle \varphi \rangle}{\underset{\parallel}{\mathbb{Z}}} \xleftarrow{0} \mathbb{Z} \longleftarrow 0$$

where $\alpha(c)=1$.

where $\varphi(a)=1$

$$\begin{aligned} \delta_1(\varphi)(c) &= \varphi(\partial_2 c) \\ &= \varphi(2a) = 2 \\ \Rightarrow \delta_1(\varphi) &= 2\alpha \end{aligned}$$

$$H^*:\quad \mathbb{Z}/2 \qquad 0 \qquad \mathbb{Z}$$