Lecture 2: Cohomology: Examples and Properties.

Last time: X space

$$\cdots \longrightarrow C_{n+1} \longrightarrow C_n \xrightarrow{\partial} C_{n-1} \longrightarrow \cdots \qquad C_n = C_n(X;\mathbb{Z})$$

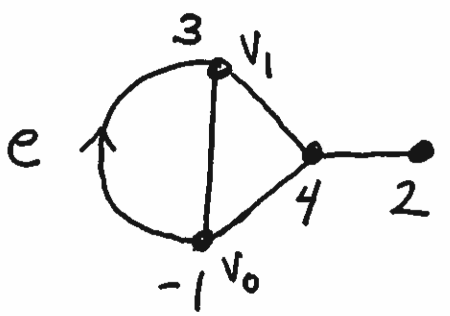
$$\cdots \longleftarrow C^{n+1} \longleftarrow C^n \xleftarrow{\delta} C^{n-1} \longleftarrow \cdots \qquad C^n = \mathrm{Hom}(C_n, G)$$

$$\boxed{\delta(\varphi) = \varphi \circ \partial} \qquad = \left\{ \begin{array}{l} \text{fn's from } n\text{-simp} \\ \text{to } G \end{array} \right\}$$

Cohomology: $H^n(X;G) = \ker \delta_n \big/ \operatorname{im} \delta_{n-1}$

Today: Examples, properties, Universal Coeff Theorem.

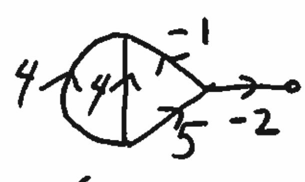
Example 1: X a graph, $G = \mathbb{Z}$



$$0 \longleftarrow \underset{\substack{\| \\ \text{fns}(\text{edges} \mapsto \mathbb{Z})}}{C^1} \xleftarrow{\delta} \overset{\substack{\text{fns}(\text{vert} \mapsto \mathbb{Z}) \\ \|}}{C^0} \longleftarrow 0 \qquad \varphi \in C^0$$

$$\delta(\varphi)(e) = \varphi(\partial e) = \varphi(v_1 - v_0) = \varphi(v_1) - \varphi(v_0) = 4$$

$= $ difference of φ at endpts.



$\delta\varphi$

[(Rate) of change, like a derivative.]

so

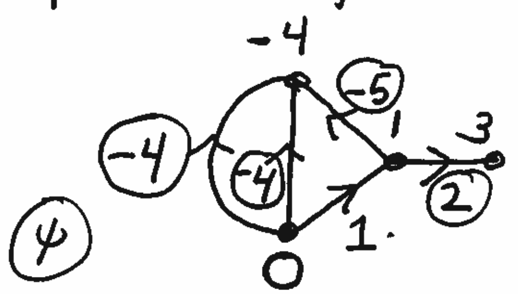
$\operatorname{Ker} \delta = $ const fns on X^0 $\Rightarrow H^0 = \mathbb{Z}$

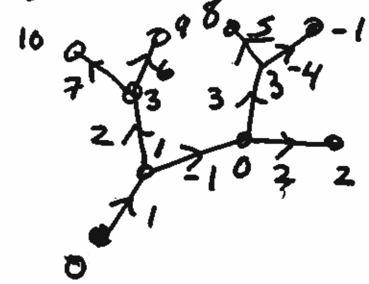
In general, for arb. X:

$$H^0(X) = \prod_{\text{components of } X} \mathbb{Z}$$

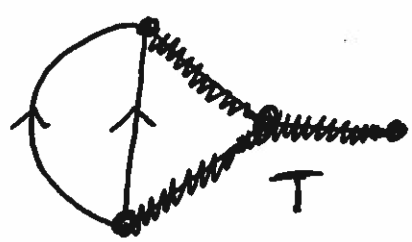
[Contrast: $H_0(X) = \bigoplus_{\text{comps}} \mathbb{Z}$]

Now $\operatorname{im} \delta = \{\psi : \text{edges} \to \mathbb{Z} \mid \exists \varphi \in C^0 \text{ with } \delta\varphi = \psi\}$ is like finding a fn given its derivative; solution is unique up to const fns. For a tree Y, $H^1 = 0$:





For X, conside the maximal tree. Given $\psi \in C^1$



replace with

$$\psi' = \psi + \delta\varphi$$

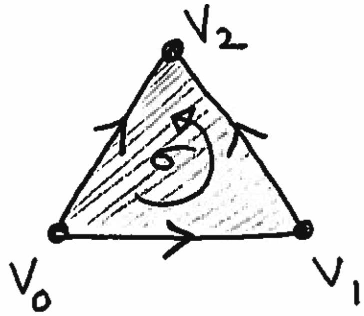
so that $\psi' = 0$ on the tree. Thus

$$H^1 = C^1 / \operatorname{im}\delta \cong \frac{\text{Subset of } C^1}{\text{vanishing on } T} \cong \mathbb{Z}^2$$

For a general graph, get

$$\Rightarrow H^1 \cong \prod_{\substack{\text{edges outside} \\ \text{max tree}}} \mathbb{Z} \quad \text{vs.} \quad H_1 = \bigoplus_{\text{same}} \mathbb{Z}$$

Ex 2:



$$\partial\sigma = [v_0, v_1] - [v_0, v_2] + [v_1, v_2]$$

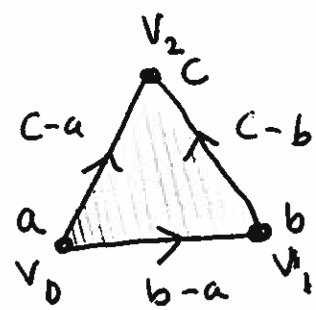
$$C^2 \xleftarrow{\delta} C^1 \ni \psi$$

$$\begin{aligned}\delta\psi(\sigma) &= \psi(\partial\sigma)\\ &= \psi([v_0,v_1]) - \psi([v_0,v_2]) + \psi([v_1,v_2])\\ &= \text{sum of values of } \psi \text{ around } \sigma\end{aligned}$$

So $\delta\psi = 0 \iff \underbrace{\psi([v_0,v_2]) = \psi([v_0,v_1]) + \psi([v_1,v_2])}$

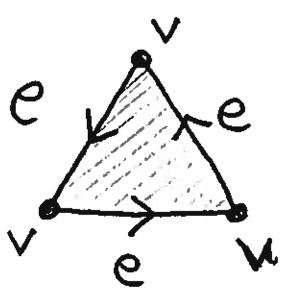
"Line integral which is path independent."

When $\psi = \delta\varphi$ for $\varphi \in C^0$ we have and in particular $\delta\psi = 0$.



Think of as a local integrability check.

Ex 3:



C_*:

$$0 \to \mathbb{Z} \xrightarrow{\times 3} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \to 0$$

dim: 2 1 0

H_*: 0 $\quad \mathbb{Z}/3$ $\quad \mathbb{Z}$

$$C^*: 0 \leftarrow \mathbb{Z} \xleftarrow{\times 3} \mathbb{Z} \xleftarrow{0} \mathbb{Z} \leftarrow 0$$

dim: 2 1 0

H^*: $\mathbb{Z}/3$ $\quad 0$ $\quad \mathbb{Z}$

Basics [Hatcher § 3.1 part 2]

Reduced version: $\tilde{H}^k$

$$\tilde{C}_*: \to C_2 \to C_1 \xrightarrow{\varepsilon} \mathbb{Z} \to 0$$
$$\tilde{C}^*: \leftarrow C^2 \leftarrow C^1 \xleftarrow{\varepsilon^*} G \leftarrow 0$$

Relative version: $H^k(X,A)$ using $C^*(X,A) = \mathrm{Hom}(C_*(X,A;\mathbb{Z}),$

Long exact seq: $A \xhookrightarrow{i} X \xrightarrow{j} (X,A)$

$$\cdots \leftarrow H^{n+1}(X,A) \xleftarrow{\delta} H^n(A) \xleftarrow{i^*} H^n(X) \xleftarrow{j^*} H^n(X,A) \xleftarrow{\delta} H^{n-1}(A) \leftarrow \cdots$$

Excision: $A \subseteq X$ closed w/ nbhd that def retracts to A.

$$\tilde{H}^n(X/A) \cong H^n(X,A)$$

[Simplical = Cellular, Mayer-Vietoris, invariant under homotopy equiv.]

[All analogous to homology case or follow from the U.C.T.]

Universal Coeff Thm (Easy version) Suppose $H_n = H_n(X;\mathbb{Z})$ are all finitely generated. Let T_n = torsion subgp of H_n $\left[H_n = \mathbb{Z}^d \oplus \underset{\text{finite}}{T_n} \right]$ Then

$$H^n(X;\mathbb{Z}) \cong H_n/T_n \oplus T_{n-1}$$

General setup: $C = \{C_n\}$ a chain cx of free ab. gps

H_n homology

$H^n(C;G)$ = cohomology of $C^n = \mathrm{Hom}(C_n, G)$

Thm: $0 \to \mathrm{Ext}(H_{n-1}(C), G) \longrightarrow H^n(C;G) \xrightarrow{h} \mathrm{Hom}(H_n, G) \to$ is exact; moreover the sequence splits, but not naturally.

Initial focus: $H^n(C;G) \xrightarrow{h} \mathrm{Hom}(H_n, G)$

Idea: $[\varphi] \in H^n(C;G)$ $\quad \varphi \in C^n = \mathrm{Hom}(C_n, G)$ $\delta\varphi = 0$

$[\alpha] \in H_n$ $\quad \alpha \in C_n$ $\partial\alpha = 0$

Set $h([\varphi])([\alpha]) = \varphi(\alpha) \in G$

Well-defined: (a) Replacing α with $\alpha + \partial\beta$ gives

$$\varphi(\alpha+\partial\beta) = \varphi(\alpha) + \varphi(\partial\beta) = \varphi(\alpha) + \underbrace{(\delta\varphi)}_{0}(\beta) = \varphi(\alpha) \checkmark$$

(b) Replacing φ with $\varphi + \delta\psi$ is OK since

$$\delta\psi(\alpha) = \psi(\partial\alpha) = \psi(0) = 0. \checkmark \qquad \blacksquare$$

Claim: h is surjective. $\underset{\text{cycles}}{Z_n} = \ker\partial_n \supseteq \operatorname{im}\partial_{n+1} = \underset{\text{boundaries}}{B_n}$

$H_n = Z_n / B_n$.

Given $\varphi_0 \in \operatorname{Hom}(H_n, G)$ want $\varphi \in H^n(C;G)$ with $h(\varphi) = \varphi_0$. Have

$$\begin{tikzcd} 0 \arrow[r] & Z_n \arrow[r, shift left, "i"] & \underset{\underset{Z_n \oplus B_{n-1}}{\cong}}{\overset{\overset{\text{Free ab}}{\downarrow}}{C_n}} \arrow[l, dashed, shift left, "\rho"] \arrow[r, shift left, "\partial"] & \overset{\overset{\text{Free ab}}{\downarrow}}{B_{n-1}} \arrow[l, dotted, shift left] \arrow[r] & 0 \end{tikzcd}$$

which splits since everything is free abelian. Use to construct ρ so that $\rho \circ i = \mathrm{id}_{Z_n}$. Define $\varphi_1 : C_n \to G$ by the composition

$$\begin{tikzcd} C_n \arrow[r, "\rho"'] \arrow[rrr, bend right, "\varphi_1"'] & Z_n \arrow[r] & H_n \arrow[r, "\varphi_0"] & G. \end{tikzcd}$$

Now $\delta\varphi_1(\alpha) = \varphi_1(\partial\alpha) = \varphi_0([\partial\alpha]) = 0$

some elt of C_{n+1}

So $\varphi_1 \in C^n(C;G)$ is a cocycle and clearly

$h(\varphi_1) = \varphi_0$. $\blacksquare$