# Lecture 33:



Goal: $X$ a CW complex, $G$ an abelian gp. $n > 0$

Then $H^n(X;G) \cong \langle X, K(G,n)\rangle$

Reduced Suspension: $X$ space w/ basepoint $x_0$

$$\Sigma X = SX \Big/ \{x_0\}\times I = X \times [-1,1] \Big/ \{x_0\}\times I \cup X \times \{-1,1\}$$

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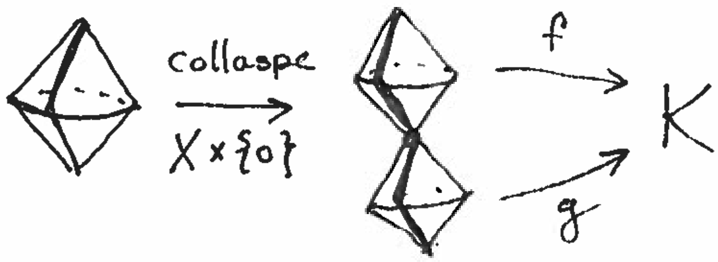
[Trying to understand why $\langle X, K(G,n)\rangle$ is a group.]

$f, g \in \langle \Sigma X, K\rangle$ (some space). Define $f+g \in \langle \Sigma X, K\rangle$

via.




$\Sigma X \vee \Sigma X$

wedge at basepoint.

Turns out this is an abelian group operation if

$X = \Sigma Y$.

The loopspace of $K$ is

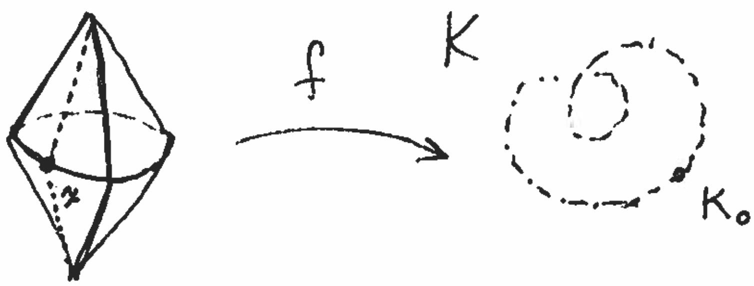
$$\Omega K = \left\{ \begin{matrix}\text{continuous}\\ \text{maps}\end{matrix} \ (I, \partial I) \longrightarrow (K, k_0) \right\}$$

with the $\underbrace{\text{compact open topology}}$. The base pt of $\Omega K$ is $\mathrm{const}_{k_0}$. [Query]

Adjoint Relation: $\langle \Sigma X, K \rangle \cong \langle X, \Omega K \rangle$

via: $(f : \Sigma X \to K) \longmapsto (x \mapsto f|_{x \times I})$

Pf: $\Sigma X$ $\xrightarrow{f}$ $K$



Now $\langle \Sigma X, K \rangle = f : X \times I \longrightarrow K$

where $(\{x_0\} \times I) \cup (X \times \partial I)$

and $\langle X, \Omega K \rangle = F : X \longrightarrow \Omega K$

$x_0 \longmapsto \mathrm{const}_{k_0}$

where $F$ gives $\bar{F} : X \times I \longrightarrow K$ via $\bar{F}(x,t) = F(x)(t)$

and $\bar{F}(X \times \partial I) = k_0$ since $F(x)$ is a loop based at $k_0$

and $\bar{F}(\{x_0\} \times I) = k_0$ since $F$ preserves basepts. ▨

Cor: $\pi_n K = \langle S^n, K\rangle = \langle S^{n-1}, \Omega K\rangle$
$$= \pi_{n-1}(\Omega K) = \pi_0(\Omega^n K)$$

So $\Omega(K(G,n))$ is a $K(G,n-1) \Rightarrow \Omega\mathbb{C}P^\infty \simeq_{h.e.} S^1$.

Concatenation of loops gives $\Omega K \times \Omega K \longrightarrow \Omega K$ (as in def of $\pi_1$). Can then define a sum in $\langle X, \Omega K\rangle$ via $(f+g)(x) = (f(x)) * (g(x))$.

This is the adjoint of the sum on $\langle \Sigma X, K\rangle$.

By same reasoning as for $\pi_1$, get that $\langle X, \Omega K\rangle$ is a group and $\langle X, \Omega^n K\rangle$ is abelian for $n \geq 2$.

$\Omega$-spectrum: A sequence of CW complexes $\{K_n\}$ together with homotopy equivalences $K_n \longrightarrow \Omega K_{n+1}$

Ex: Fix an abelian gp $G$, take $K_n = K(G,n)$ for $n>0$.

Ex: $O = \bigcup O(n)$. Bott periodicity: $O \simeq_{h.e.} \Omega^8 O$

$K_n$ = CW model of $\Omega^{(n \bmod 8)} O$. for $n \in \mathbb{Z}$

<u>Thm</u>: If $K_n$ is an $\Omega$-spectrum, then the functor $X \longmapsto h^n(X) = \langle X, K_n \rangle$ defines a reduced cohomology theory on the category of based CW complexes.

[For the O spectrum, get real K-theory.]

<u>Axioms</u>: $\underbrace{h^n : CW \longrightarrow Ab}_{\text{functors}}$ and $\delta : h^n(A) \longrightarrow h^{n+1}(X/A)$ where ($A$ subcomplex of $X$)

① homotopic maps induce the same map on $h^n$.

② Long exact seq for $(X, A)$

③ $h^n\left(\bigvee_\alpha X_\alpha\right) \cong \prod_\alpha h^n(X_\alpha)$ via the product of $p_\alpha^* : h^n(\vee_\alpha X_\alpha) \longrightarrow h^n(X_\alpha)$ where $p_\alpha$ is the "projection" $\vee_\alpha X_\alpha \longrightarrow X_\alpha$.

[Note the converse of the theorem is true as well. This is called Brown Representability]

Proof: ① Given $f: X \to Y$ get $\langle X, K_n \rangle \xleftarrow{f^*} \langle Y, K_n \rangle$ defined by $f^*(g: Y \to K_n) = g \circ f$. Clearly this only depends on the homotopy class of $f$. Also $f^*$ is a homomorphism in the gp structure coming from $K_n = \Omega K_{n+1}$.

③ $\left\{ \begin{array}{l} \text{base point pres} \\ \text{maps } \bigvee_\alpha X_\alpha \to K_n \end{array} \right\} \cong \left\{ \begin{array}{c} \text{A coll. of based maps} \\ X_\alpha \longrightarrow K_n \end{array} \right\}$

② Next time.

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Based vs. Unbased Cohomology. Given

$\tilde{h}^*$ a reduced based theory, define

$h^n(X) = \tilde{h}^n(X_+ = X \text{ w/ disjoint base pt});$

this is an <u>unreduced unbased</u> theory.