More on fiber bundles.

Suppose G is a topological group acting on a space X.

A **fiber bundle with fiber F and structure group G** is a map $p: E \to B$ together with a collection of homeom.

$\{\varphi : p^{-1}(U) \to U \times F\}$ for certain $U \subseteq B$ where:

1) The U cover B.

2) Each
$$\begin{tikzcd} p^{-1}(U) \arrow[rr, "\varphi"] \arrow[dr, "p"'] & & U \times F \arrow[dl, "\mathrm{proj}_U"] \\ & U & \end{tikzcd}$$
commutes

3) If (U, φ) and (V, ψ) are charts, $\exists$ a continuous map $\theta : U \cap V \to G$ so that $\forall\, x \in U \cap V$ and $f \in F$ have
$$(x, \theta(x) \cdot f) = \varphi(\psi^{-1}((x, f)))$$

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4) If (U, φ) is a chart and $V \underset{\text{open}}{\subseteq} U$ so is $(V, \varphi|_V)$.

5) The collection (U, φ) is maximal with respect to (1-4).

6) G acts effectively on F, i.e. $G \hookrightarrow \mathrm{Homeo}(F)$.

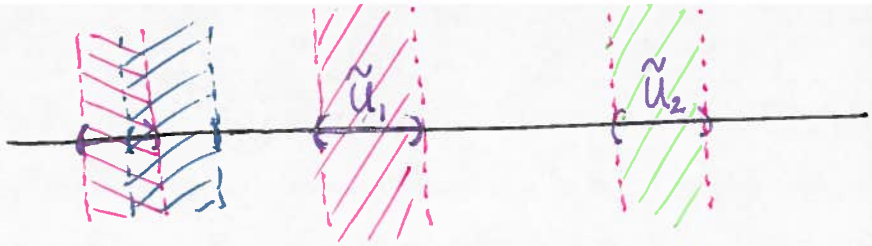
Principal bundles: G = any top gp, $F = G$ acted on by left trans.

Construction: $\rho : \pi_1 B \to G$ $\quad$ $\tilde{E} = \tilde{B} \times G$ acted on by $\pi_1 B$ via $\gamma \cdot (\tilde{b}, g) = (\gamma \cdot \tilde{b}, \rho(\gamma) \cdot g)$

$$E = \pi_1 B \backslash \tilde{B} \times G$$

$$p : E \to B$$
$$[(\tilde{b}, g)] \mapsto \pi(\tilde{b})$$

U evenly covered.



$\tilde{U}_1$ $\tilde{U}_2$ $\tilde{B}\times G$

U

$$\tilde{B}\times G \xrightarrow{q} E \xrightarrow{p} B$$

coor to $\tilde{U}_1$ coor to $\tilde{U}_2$

$$U\times G \xleftarrow{\varphi_1} p^{-1}(U) \xrightarrow{\varphi_2} U\times G$$

Take $\theta: U \to G$ to be $\rho(\gamma)$ where $\gamma\in\pi_1 B$ sends $\tilde{U}_2$ to $\tilde{U}_1$. Then $(x, \theta(x)\cdot g) = \varphi_1\circ\varphi_2^{-1}(x, g)$.

Allows us to build <u>many</u> bundles.

$B =$  $\quad \pi_1 B \twoheadrightarrow F_2 = \langle x, y\rangle$

Now pick $X, Y \in SU(2)$ to get a 5-manifold $E_{X,Y} \to B$.

<u>Note</u>: ① This construction gives a <u>flat</u> principal bundle (θ's are locally constant).

② <u>Generalization</u>: G acts on F, get bundle where $\gamma\in\pi_1 B$ acts by $(\tilde{b}, f) \mapsto (\gamma\cdot\tilde{b}, \rho(\gamma)\cdot f)$.

$$\tilde{B}\times F \,/\, \pi_1 B \;\longrightarrow\; B$$

If G is discrete, a principal G-bundle is a regular covering space corresponding to a homomorphism $\pi_1 B \to G$.

<u>Pull backs:</u> $p: E \to B$ fiber bundle. Given $f: A \to B$, define $f^*(E) = \underbrace{\{(a,e)}_{\text{in } A \times E} \mid f(a) = p(e)\}$. HW: The map $\pi_A : f^*(E) \to A$ is a fiber bundle with the same fiber and structure gp as $E \to B$.

<u>Universal Bundles:</u> If G is a topological group, there exists a principal G-bundle $EG \to BG$ so that $\forall$ CW's X we have

$$[X, BG] \cong \begin{array}{c}\text{Isomorphism classes}\\ \text{of principal } G\text{-bundles}\\ \text{over } B\end{array}$$

$$(f : X \to BG) \longmapsto f^*(EG)$$

If G has the discrete topology, then $BG = K(G,1)$ and $EG \to BG$ is as in the 1st proof of the existence of $K(G,1)$'s.

Homology with local coefficients. (§3.H in Hatcher)

Suppose $p: E \to B$ is a principal G-bundle with G discrete and abelian. [Note covering map.]

$$C_n(B;E) = \left\{ \begin{array}{l} \text{finite sums } \sum n_i \sigma_i \text{ where} \\ \sigma_i : \Delta^n \to B \text{ is a singular} \\ \text{simplex, and } n_i : \Delta^n \to E \\ \text{is a lift of } \sigma_i \end{array} \right\}$$

If m_i and n_i are lifts of the same σ_i then define $(m_i + n_i) : \Delta^n \to E$ by $(m_i + n_i)(s) = m_i(s) + n_i(s)$ and $n_i\sigma_i + m_i\sigma_i = (n_i + m_i)\sigma_i$. Gives group structure to $C_n(B;E)$.

If $E = B \times G$ with $p = \pi_B$ then $C_n(B;E) = C_n(B;G)$

There is a $\partial : C_n(B;E) \to C_{n-1}(B;E)$ given by

$$\partial(n_i\sigma_i) = \sum (-1)^j n_i|_{[v_0,\dots,\hat{v}_j,\dots,v_n]} \, \sigma_i|_{[v_0,\dots,\hat{v}_j,\dots,v_n]}$$

and $\partial^2 = 0$ as usual.

Poincaré duality revisited: M closed n-mfld

$$\begin{array}{ccl} M_{\mathbb{Z}} & = & \{ \alpha_x \in H_n(M, M \setminus \{x\}; \mathbb{Z}) \} \\ \downarrow & & \quad \downarrow \\ M & & \quad x \end{array}$$

Principal $\mathbb{Z}$-bundle over M.

For M closed (orientable <u>or not</u>) have

$$H_n(M; M_{\mathbb{Z}}) = \mathbb{Z}$$

a fundamental class $[M]$ is a generator for this.

<u>Thm</u>: M nonorientable closed n-manifold. Then
$H^k(M; \mathbb{Z}) \cong H_{n-k}(M; M_{\mathbb{Z}})$ and $H^k(M; M_{\mathbb{Z}}) \cong H_{n-k}(M; \mathbb{Z})$

Comes from $R = \mathbb{Z}[i]$ where complex conj.
corresponds to going around non-orientable loops.