Lecture 30: Fiber bundles

<u>HLP</u>: $p: E \to B$ has HLP with respect to X if given $g_t: X \times I \to B$ and a lift $\tilde{g}_0 : X \to B$, there exists a lift $\tilde{g}_t : X \times I \to B$ extending $\tilde{g}_0$.

A <u>fibration</u> $p: E \to B$ has HLP w.r.t all X.

A <u>fiber bundle</u> $p: E \to B$ with fiber F has the prop. that B is covered by open sets U with

$$\begin{tikzcd} p^{-1}(U) \arrow[rr, "h", "\cong"'] \arrow[dr, "p"'] & \arrow[d, phantom, "\circlearrowleft"] & U \times F \arrow[dl, "\mathrm{proj}_U"] \\ & U & \end{tikzcd}$$

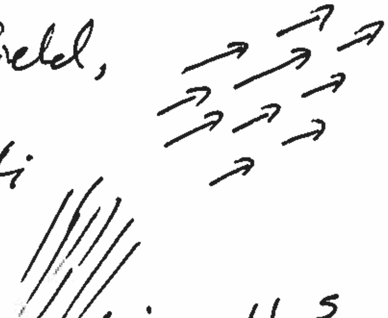
<u>Ex</u>: $S^1 \longrightarrow S^{2n+1} \longrightarrow \mathbb{C}P^n$ [Gen. Hopf bundle]

$$\begin{tikzcd} S^{2n+1} = \{ z \in \mathbb{C}^{2n} \mid \sum |z_i|^2 = 1 \} \arrow[d, "p"'] \\ \mathbb{C}P^n = \mathbb{C}^n \setminus \{0\} / \mathbb{C}^* \end{tikzcd}$$

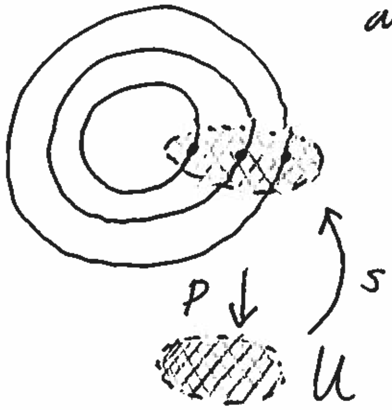
S^1 acts freely on S^{2n+1} by $\lambda \cdot z = \lambda z$

Orbits are exactly the fibers of p.

Generated by some nowhere 0 vector field, so locally fibers are a stack of spaghetti



To see product structure, find a section $U \xrightarrow{s} S^{2n+1}$ and then use the action:



$$U = \{ [z_0 : \cdots : z_n] \mid z_0 \neq 0 \}$$

$$s(U) = \{ z \in S^{2n+1} \mid z_0 \in \mathbb{R}_+ \}$$

$$s([z_0 : \quad : z_n]) = \frac{|z_0|}{z_0 |z|} z \quad \text{where } z = (z_0, \ldots, z_n)$$

$$|z| = \sqrt{\sum |z_i|^2}$$

Define $h^{-1} : U \times S^1 \longrightarrow p^{-1}(U)$ by

$$([z], \lambda) \longmapsto \lambda\, s([z])$$

It's inverse is $h(z) = (p(z), z_0/|z_0|)$ so we've proved the local product structure.

<u>Thm</u>: A fiber bundle $p : E \to B$ where B is <u>paracompact</u> is a fibration.

every open cover has a locally compact refinement.

<u>Cor</u>: $\pi_3 S^2 = \mathbb{Z}$, gen by Hopf map $S^3 \to S^2$.

<u>Pf</u>: The Hopf bundle $S^1 \to S^3 \to S^2$ gives a long exact sequence:

$$\to \pi_n S^1 \to \pi_n S^3 \xrightarrow{P_*} \pi_n S^2 \to \pi_{n-1} S^1 \to$$

and so $\pi_n S^3 \cong \pi_n S^2$ for $n \geq 3$. ▨

<u>Cor</u>: $\mathbb{C}P^\infty$ is a $K(\mathbb{Z}, 2)$.

<u>Fun Fact</u>: S^2 and $S^3 \times \mathbb{C}P^\infty$ has same π_* <u>NOT HOMOTOPY EQUIV</u>

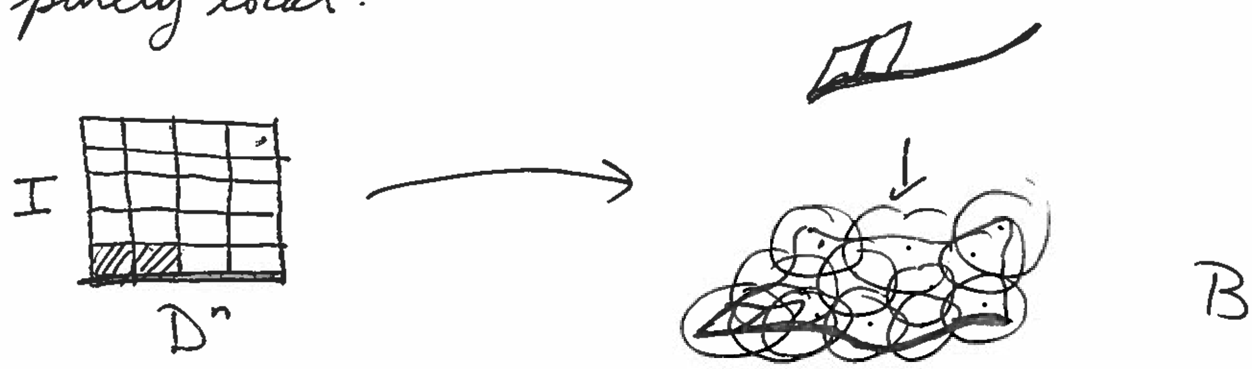
<u>Pf</u>. Previous construction yields a fiber bundle

$$S^1 \to S^\infty \to \mathbb{C}P^\infty.$$

▨

<u>Thm</u>: A fiber bundle has H.L.P. with respect to all CW complexes X. [For long exact seq, only really need HLP with respect to D^n.]

Pf is purely local:



cover by finitely many product ⊂

[Bott Periodicity:]

Fiber bundle: $O(n-1) \longrightarrow O(n) \longrightarrow S^{n-1}$

$$A \longmapsto Ae_1$$

So $\pi_* O(n)$ is at least as complicated as $\pi_*(S^m)$. By long exact seq of a fibration $O(n-1) \hookrightarrow O(n)$ gives an isom on π_i for $i < n-2$.

Thus $\pi_i O(n)$ is independent of n for large n, call it $\pi_i O$. ← or take $O = \bigcup O(n)$. Turns out to only depend on i mod 8:

i mod 8	0	1	2	3	4	5	6	7
$\pi_i O$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	0	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$

This is called Bott periodicity.

Some details: $O(n) = \{A \in GL_n\mathbb{R} \mid AA^t = I\}$

$$O(1) \leq O(2) \leq O(3) \leq O(4) \leq \cdots \qquad \begin{pmatrix} A & \begin{smallmatrix}0\\0\\0\end{smallmatrix} \\ \begin{smallmatrix}0&0&0\end{smallmatrix} & 1 \end{pmatrix}$$

$O(1) = \mathbb{Z}/2\mathbb{Z}$, $O(2) = \mathbb{Z}/2\mathbb{Z} \times S^1$, $O(3) = \mathbb{R}P^3 \amalg \mathbb{R}P^3$

Always two comps, corr to $\det = \pm 1$.

Theme: Infinite dim'l spaces can have simpler π_* than finite-dim'l ones.

Fact: There is no simply-connected finite CW complex where all of π_* is known, other than contractible ones.

Stable homotopy groups: Via suspension, have

$$\pi_i S^n \longrightarrow \pi_{i+1} S^{n+1}$$

which is an isom for $i < 2n-1$. Define

$$\pi_i^s = \pi_{i+n} S^n \quad \text{for } n > i+1$$

i	0	1	2	3	4	5	6	7	8		
π_i^s	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/24$	0	0	$\mathbb{Z}/2$	$\mathbb{Z}/240$	$\mathbb{Z}/2 \times \mathbb{Z}/2$	$\cdots$	$\mathbb{Z}/504$
	id	$\eta: S^3 \to S^2$		$S^7 \xrightarrow[\nu]{} S^4$				$S^{15} \longrightarrow S^8$			