Lecture 31: Stable homotopy groups

Last time: $O(1) \subseteq O(2) \subseteq O(3) \subseteq \cdots \qquad O = \bigcup_n O(n)$

$i \bmod 8$	0	1	2	3	4	5	6	7
$\pi_i O$ $=$ $\pi_i O(n)$, $n \geq i+3$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	0	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$

Bott periodicity

$\pi_i U$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$
$\pi_i Sp$	$\cancel{0}$	$\cancel{0}$	0	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	0	$\mathbb{Z}$

$\mathbb{R} \subseteq \mathbb{C} \subseteq \mathbb{H}$

Theme: Large ∞-dimensional spaces can have simpler π_* than finite dim'l ones.

Note: There is no simply connected finite CW complex all of whose π_* are known, except contractible ones.

$$S^n \text{ vs. } S^\infty \qquad \mathbb{C}P^n \text{ vs. } \mathbb{C}P^\infty$$

Stable Homotopy Groups: X an $(n-1)$ connected space, then $\pi_i X \xrightarrow[S]{} \pi_{i+1} SX$ is an isomorphism

$$(S^i \to X) \longmapsto (\underset{\substack{\| \\ S^{i+1}}}{SS^i} \to SX) \quad \text{for } i < 2n-1$$

In particular, SX is n-connected.

X any space

$$\pi_i X \xrightarrow{S} \underbrace{\pi_{i+1}(SX)}_{\text{0-conn}} \xrightarrow{S} \underbrace{\pi_{i+2}(S^2X)}_{\text{1-conn}} \xrightarrow{S} \underbrace{\pi_{i+3}(S^3X)}_{\text{2-conn}} \to$$

$$\cdots \to \pi_{i+k}(\underbrace{S^k X}_{(k-1)\text{-connected}}) \underset{\cong}{\to} \pi_{i+k+1}(S^{k+1}X).$$

where $i+k < 2k-1$, i.e. $i < k-1$.

So eventually these are all $\cong$. Denote the limiting group $\pi_i^S X$, called the i^{th} stable homotopy group of X.

Special Case: $X = S^0$ so $\pi_i^S S^0 = \pi_{i+n} S^n$ for $n > i+1$

π_i^S - stable i-stem [Appears in all sorts of geometric problems...]

Thm (Serre) π_i^S is always finite for $i > 0$.

i	0	1	2	3	4	5	6	7
π_i^s	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/24$	0	0	$\mathbb{Z}/2$	$\mathbb{Z}/240$
	id	$\eta: S^3 \to S^2$ hopf		$\nu: S^7 \to S^3$				$S^{15} \xrightarrow{\sigma} S^8$

$(S^7 \subseteq \mathbb{H}^2) \longrightarrow \mathbb{HP}^1 \cong S^3$

i	8	9	10	11	12
π_i^s	$(\mathbb{Z}/2)^2$	$(\mathbb{Z}/2)^3$	$\mathbb{Z}/6$	$\mathbb{Z}/504$	0

Composition

<u>Product:</u> $\pi_i^s \times \pi_j^s \longrightarrow \pi_{i+j}^s$

$[f] \quad [g] \qquad g \circ f : S^{n+i} \longrightarrow S^{n-j}$

$f : S^{n+i} \to S^n$

$g : S^n \longrightarrow S^{n-j}$

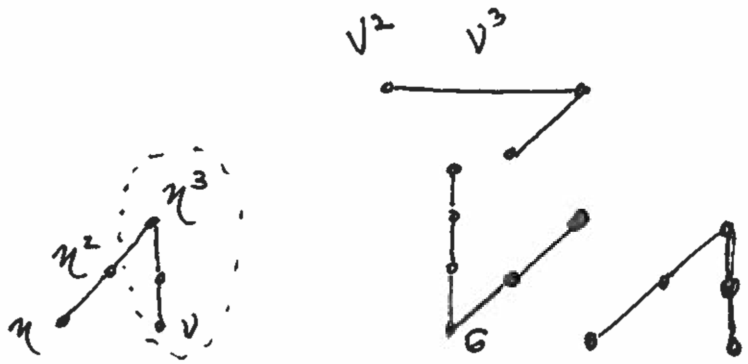
<u>Thm:</u> $\pi_*^s = \bigoplus_{i=0}^{\infty} \pi_i^s$ is a graded commutative ring; i.e. $\alpha \in \pi_i^s$, $\beta \in \pi_j^s$ then $\alpha\beta = (-1)^{ij}\beta\alpha$.

<u>Pf:</u> See text.

<u>Ex:</u> η^2 gen π_2^s

To see patterns, work prime by prime

$${}_p\pi_i^s = \begin{matrix}\text{p-part}\\ \text{of } \pi_i^s\end{matrix} = \pi_i^s \Big/ \left\langle g \in \pi_i^s \;\middle|\; \begin{matrix}\text{order}(g)\\ \text{is coprime to } p\end{matrix} \right\rangle$$



random seeming stuff

teeth $\longrightarrow$

[can do for other primes; the "p-adic ruler"]

1	2	3	4	5	6	7	8	9	10
$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/8$	0	0	$\mathbb{Z}/2$	$\mathbb{Z}/16$	$(\mathbb{Z}/2)^2$	$(\mathbb{Z}/2)^3$	

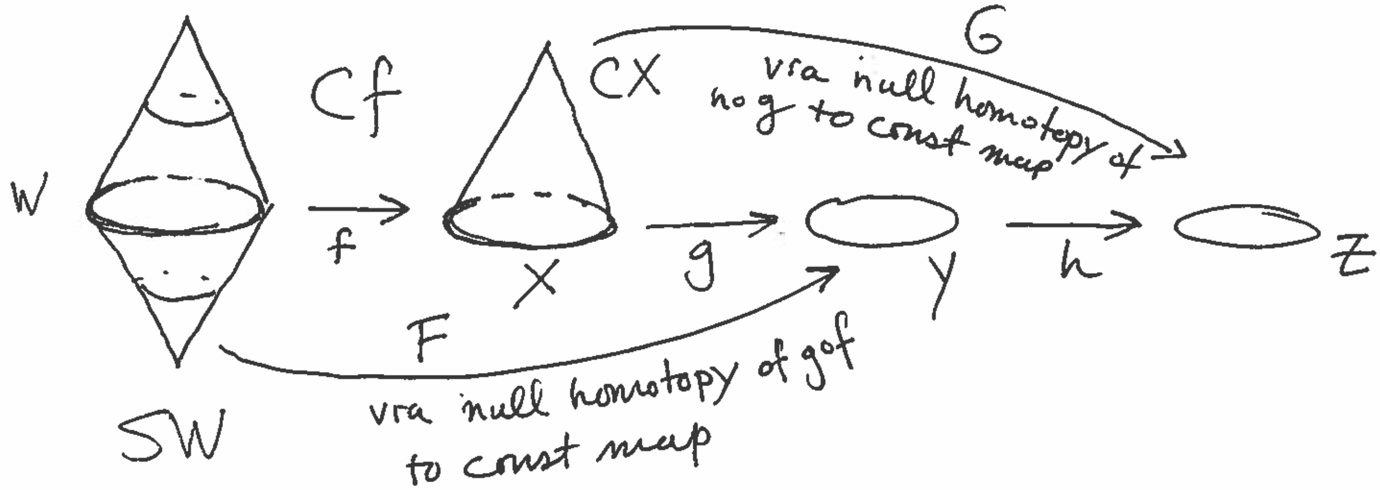
Many products are 0, but this <u>also</u> lets us construct additional elements: Suppose fg and hg are both 0

$$\begin{tikzcd} S^{i+j+k+l} \arrow[r, "f"', "\pi_i^s"] \arrow[rr, bend right] & S^{j+k+l} \arrow[r, "g"', "\pi_j^s"] \arrow[rr, bend right] & S^{k+l} \arrow[r, "h"', "\pi_k^s"] & S^{l} \end{tikzcd}$$

More generally

$$W \xrightarrow{f} X \xrightarrow{g} Y \xrightarrow{h} Z$$ with $g \circ f$ and $h \circ g$ both null-homotopic.

The Toda bracket $\langle f, g, h \rangle : SW \to Z$ is



Toda bracket is $G \circ Cf$ on C_+W and $h \circ F$ on C_-W. Not uniquely determined

<u>Ex</u>: In stable homotopy gp case, get

$$\langle f, g, h \rangle \in \pi^s_{i+j+k+1}$$

$\langle \nu, \eta, \nu \rangle$ is the unlabled gen in ${}_2\pi^s_8$