Lecture 16: Second proof of Poincaré Duality

Thm: M^n closed connected with a PL triangulation $\mathcal{T}$. Then

ⓐ $H^k(M;\mathbb{F}_2) \cong H_{n-k}(M;\mathbb{F}_2)$

ⓑ $$\begin{aligned} H_k(M;\mathbb{F}_2) \times H_{n-k}(M;\mathbb{F}_2) &\longrightarrow \mathbb{F}_2 \\ \alpha \quad\quad\quad \beta \quad\quad &\quad\quad \alpha \cap \beta \end{aligned}$$ is nondegenerate.

Here, $\alpha \cap \beta$ can be defined geometrically or by

$$\alpha \cap \beta = \left(D^{-1}(\alpha) \cup D^{-1}(\beta)\right)([M])$$

where $D: H^*(M) \xrightarrow{\cong} H_*(M)$.

$$\varphi \longmapsto [M] \cap \varphi$$

Kinds of manifolds:

TOP: topological mflds and cont. maps.

PL: Manifolds w/ PL triangulations and PL maps.

DIFF: Smooth mflds and smooth maps.

In general, neither are injective or surjective.

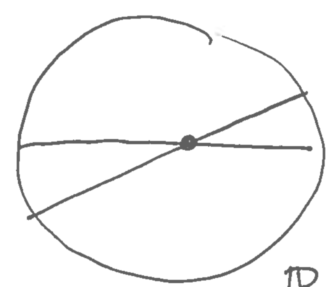
Are $\cong$ for dim 1, 2, 3.

Use homology cap product to compute cup product.

<u>Thm</u>: $H^*(\mathbb{RP}^n; \mathbb{F}_2) \cong \mathbb{F}_2(\alpha)\big/_{\alpha^{n+1}=0}$ $\quad |\alpha| = 1$

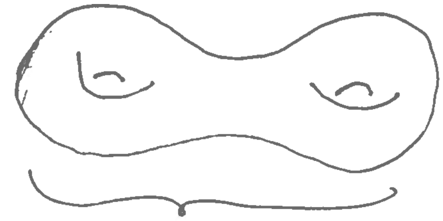
<u>Pf</u>: Let α_k be the none-zero elt of $H^k(\mathbb{RP}^n; \mathbb{F}_2)$. Since $\mathbb{RP}^{n-1} \hookrightarrow \mathbb{RP}^n$ gives an $\cong$ on H^* for $* < n$, need only check that $\alpha_{n-1} \cup \alpha_1 = \alpha_n$. The P.D. of α_k is the non-zero elt of $H_{n-k}(\mathbb{RP}^n; \mathbb{F}_2)$ which can be rep. by $\mathbb{RP}^{n-k} \hookrightarrow \mathbb{RP}^n$. Putting these in general pos. we see that

$$\mathbb{RP}^1 \cap \mathbb{RP}^{n-1} = 1 \text{ pt}$$



and so $\alpha_{n-1} \cup \alpha_1 = \alpha_n$. ▨

<u>Inductive proof of Poincaré</u>:



Can't subdivide into closed mflds.

Also P.D. fails for $\mathbb{R}^n$:

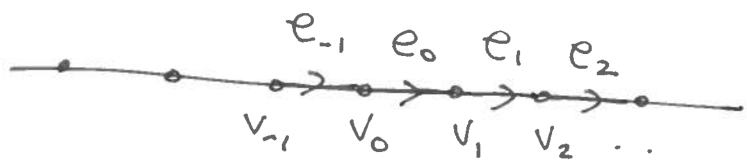
$$\underset{\mathbb{Z}}{H^0(\mathbb{R}^n; \mathbb{Z})} \not\cong \underset{0}{H_n(\mathbb{R}^n; \mathbb{Z})}$$

Cohomology with compact supports.

Simplicial version: X Δ-complex, locally finite.

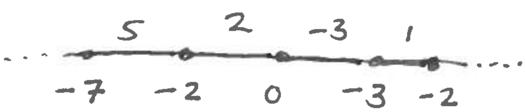
$C^i_c(X)$ = cochains taking non-zero values on only finitely many simplices. [Subcomplex because of]

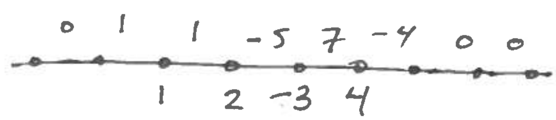
Ex: $\mathbb{R}$



ker δ_0 in C^0 is gen by $\varphi : v_i \to 1$ $\supseteq$ ker $\delta_0 |_{C^0_c} = \{0\}$ $\Rightarrow H^0_c(\mathbb{R}) = 0$

im δ_0 in C^1 is everything $\supseteq$ im $\delta_0 |_{C^1_c}$ only contains things with sum $= 0$. $\Rightarrow H^1_c(\mathbb{R}) = \mathbb{Z}$





Note: Poincaré Duality holds here for H^k_c!

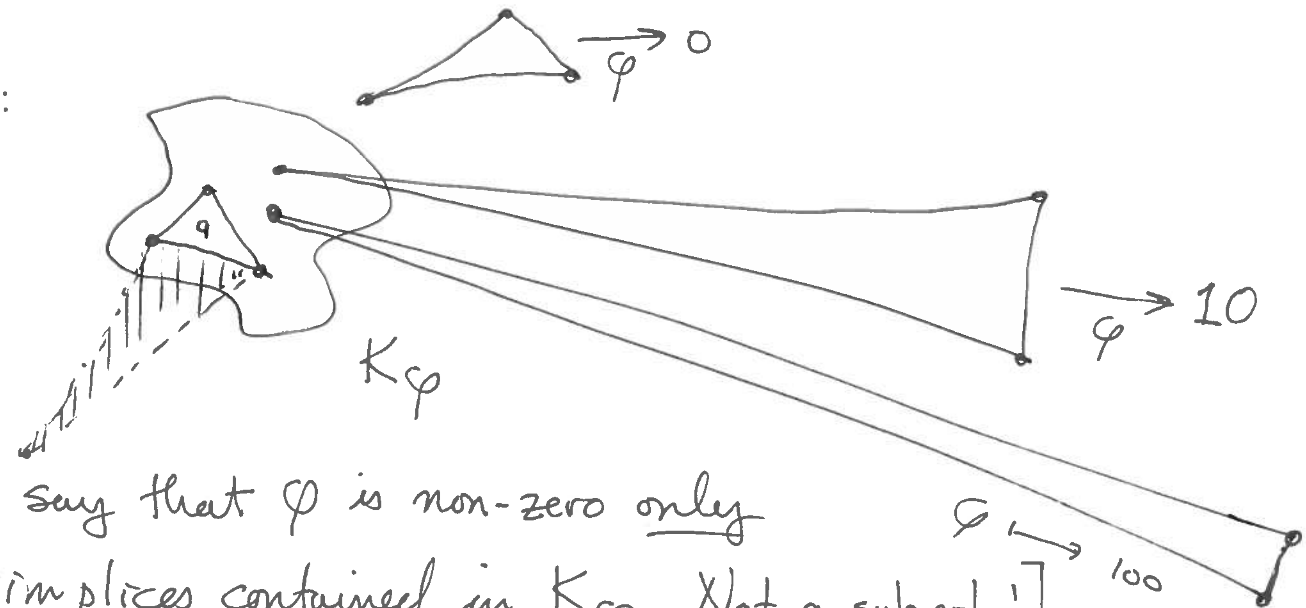
de Rahm: $\Omega^k(M)$ smooth k-forms

$$\Omega^k(M) \supseteq \Omega^k_c(M) = \left\{ \alpha \in \Omega^k(M) \;\middle|\; \exists K^{\text{cpt}} \subseteq M \text{ with } \alpha = 0 \text{ in } M \setminus K \right\}$$

<u>Singular</u>: $C^i_c(X) = \{\varphi \in C^i(X) \mid \exists K_\varphi^{\text{cpt}} \subseteq X$ with $\varphi(\sigma) = 0 \;\forall\, \sigma : \Delta_k \to X \setminus K\}$

[Subcomplex so get] $\Rightarrow H^i_c(X)$.

<u>Issue</u>:

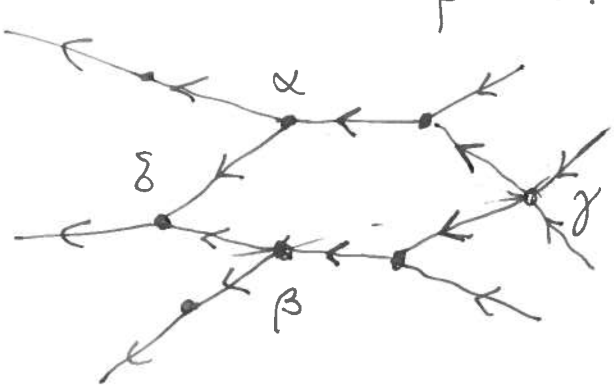


[Want to say that φ is non-zero <u>only</u> on simplices <u>contained</u> in K_φ. Not a subcplx!]

<u>Goal</u>: $H^i_c(X) = \varinjlim \underbrace{H^i(X \mid K)}_{[H^i(X,\, X \setminus K)]}$

[Query: How many have seen direct and/or inverse limits?]

I = partially ordered set where $\forall \alpha, \beta \in I \;\exists \gamma$ with $\alpha \leq \gamma \quad \beta \leq \gamma$.



[Directed graph w/ no directed cycles. Any α, β are "below" some other vertex.]

<u>Groups</u>: G_α for each $\alpha \in I$

$$f_{\alpha\beta} : G_\alpha \to G_\beta \text{ for each } \alpha \le \beta$$

If $\alpha \le \beta \le \gamma$ have $f_{\alpha\gamma} = f_{\beta\gamma} \circ f$

[Called a directed system of groups.]

<u>Direct Limit</u>:

$$\varinjlim G_\alpha = \coprod_\alpha G_\alpha \Big/ \begin{array}{l} a \in G_\alpha \sim b \in G_\beta \\ \text{if } \exists\, \gamma \text{ with } \alpha \le \gamma,\ \beta \le \gamma \\ \text{and } f_{\alpha\gamma}(a) = f_{\beta\gamma}(b) \text{ in } G_\gamma. \end{array}$$

[Check that this an equiv. relation. Can add:]

Set $[a] + [b] = [f_{\alpha\gamma}(a) + f_{\beta\gamma}(b)]$ where $\alpha, \beta \le \gamma$.

[Alternatively] $= \bigoplus_\alpha G_\alpha \Big/ \begin{array}{l} \text{subgp gen by} \\ a - f_{\alpha\beta}(a) \ \forall a \in G_\alpha \\ \forall \beta \ge \alpha. \end{array}$

<u>Ex</u>: $I = \mathbb{Z}$

$$G_\alpha = \mathbb{Z}^\alpha \qquad \varinjlim G_\alpha = \bigoplus_{\alpha=1}^{\infty} \mathbb{Z}$$

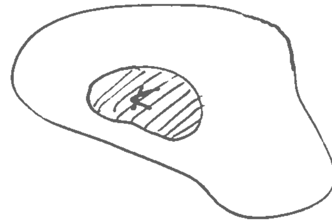
$$f_{\alpha\beta} : \mathbb{Z}^\alpha \to \mathbb{Z}^\beta$$

$$(x_1, \ldots, x_\alpha) \mapsto (x_1, \ldots, x_\alpha, 0, \ldots, 0)$$

X top space

$I = \{\text{cpt subsets of } X\}$ $\quad \leq$ = containment of subsets

$G_K = H^n(X \mid K)$



K'

$$\begin{array}{c}(X, X \setminus K') \\ \downarrow \\ (X, X \setminus K)\end{array}$$

If $K \subseteq K'$ then have

$$H^n(X \mid K) \longrightarrow H^n(X \mid K')$$

[Typically not injective or surjective.]

<u>Prop</u>: $H^n_c(X) \cong \varinjlim_{K \text{ cpt}} H^n(X \mid K)$

<u>Pf</u>: First note that $H^n(X \mid K) \longrightarrow H^n_c(X)$ since $\varphi \in C^n(X \mid K)$ vanishes on any σ in $X \setminus K$.
[Compatible with inclusions, so get] $\varinjlim H^n(X \mid K) \to H^n_c(X)$

<u>Surjective</u>: $[\varphi] \in H^n_c(X)$ is in the image of $H^n(X \mid K_\varphi)$

<u>Injective</u>: If $[\varphi] \in H^n(X \mid K)$ is 0 in $H^n_c(X)$,

then $\varphi = \delta \psi$ where ψ is supported on $K_\psi \supseteq K_\varphi$

In particular, $[\varphi] = 0$ in $H^n(X \mid K_\psi)$. $\blacksquare$