Lecture 20: Higher homotopy groups

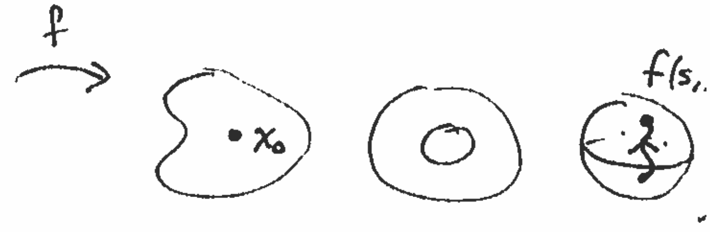
Fix a basept s_0 in S^n. X a top. sp.

$\pi_n(X, x_0)$ = Homotopy classes of maps $(S^n, s_0) \to (X, x_0)$

[All maps in the homotopy must be of this form.]

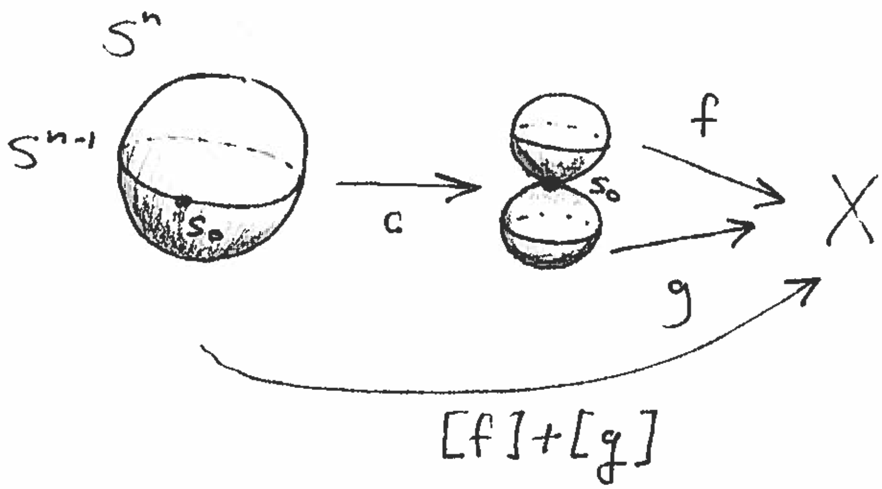
$\pi_1(X, x_0)$ = fund. gp.

$\pi_0(X, x_0) = \{ S^0 = \overset{s_0}{\bullet} \quad \overset{s_1}{\bullet}$



= set of path components of X.

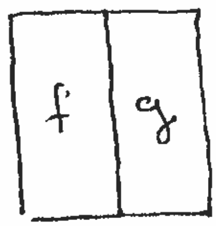
π_n for $n \geq 2$: Higher homotopy gps.



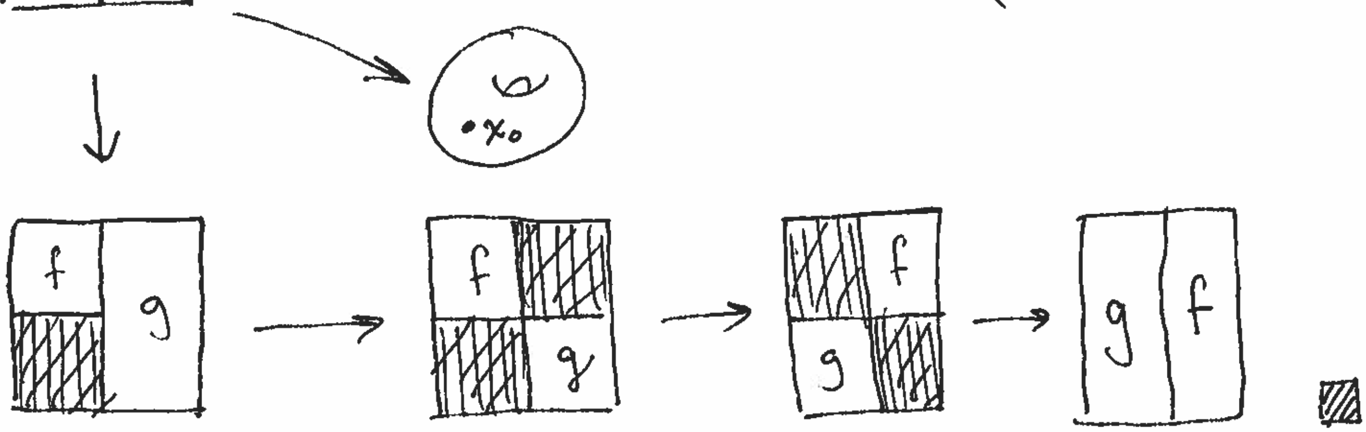
Prop: π_n is abelian for $n \geq 2$.

[Blather about how powerful yet hard to compute these are.]

Pf: $\pi_n = $ hom classes of maps $(I^n, \partial I^n) \to (X, x_0)$



$$([f]+[g])(s_1,\dots,s_n) = \begin{cases} f(2s_1, s_2, \dots, s_n) & s_1 \le 1/2 \\ g(2s_1-1, s_2, \dots, s_n) & \end{cases}$$

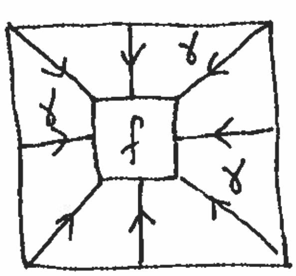


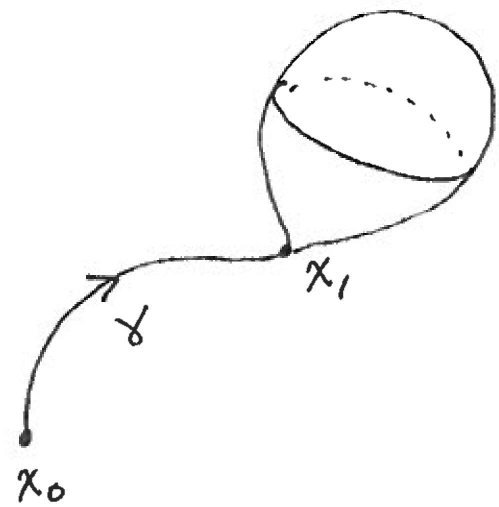
Prop: If X is path conn. then $\pi_n(X, x_0)$ is independent of x_0.

Pf. Let γ be a path from x_0 to x_1

$f : (I, \partial I) \to (X, x_1)$

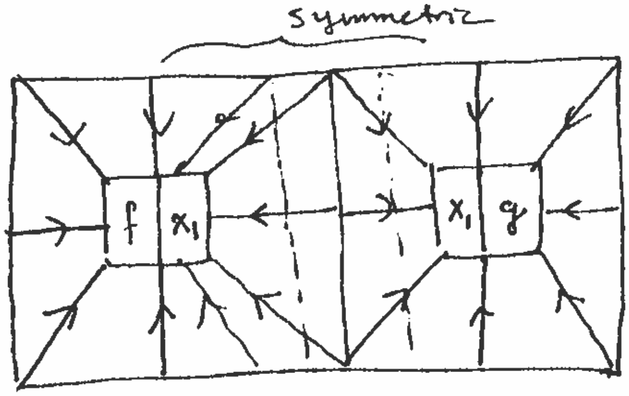
$\gamma f : (I, \partial I) \to (X, x_0)$





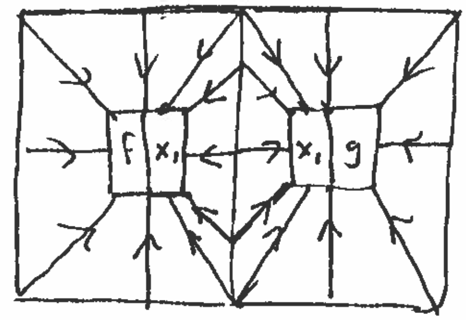
Induces a homomorphism

$\pi_1(X, x_1) \to \pi_1(X, x_0)$ since

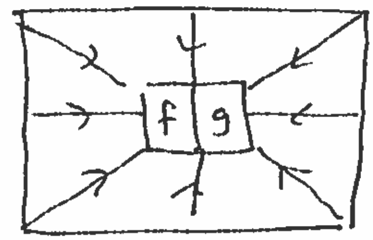


$[\gamma f] + [\gamma g]$

$\longrightarrow$



$\downarrow$



$\gamma(f+g)$

$\blacksquare$

Also have $\gamma^{-1}: \pi_n(X, x_0) \to \pi_n(X, x_1)$ which is the inverse isomorphism.

[Sometimes drop the base pt.]

<u>π_n is a functor:</u> $\varphi : (X, x_0) \longrightarrow (Y, y_0)$

induces $\pi_n(X, x_0) \xrightarrow[\varphi_*]{} \pi_n(Y, y_0)$ by

$$\begin{tikzcd}(S^n, s_0) \arrow[r, "f"'] \arrow[rr, bend right] & (X, x_0) \arrow[r, "\varphi"'] & (Y, y_0)\end{tikzcd}$$

$\varphi_*[f] = \varphi \circ f.$

$\boxed{\text{Note. } \varphi_* \text{ only depends on the } \underline{\text{based}} \text{ homotopy class of } \varphi.}$

Properly, let $Top^\cdot = \left\{ \begin{array}{l} \text{Category of } \underline{\text{based}} \text{ top. spaces} \\ \text{and } \underline{\text{base pt pres.}} \text{ cont. maps} \end{array} \right\}$

$\downarrow$

$Top = \left\{ \begin{array}{c} \text{topological spaces} \\ \text{and cont maps} \end{array} \right\}$

$\pi_0 : \text{Top}^\bullet \longrightarrow \text{Sets}$ $\qquad$ $\pi_n : \text{Top}^\bullet \longrightarrow$ Abelian Groups

$\pi_1 : \text{Top}^\bullet \longrightarrow \text{Groups}$

Covariant Functors

[HW] If $\varphi : (X, x_0) \longrightarrow (Y, y_0)$ is a homotopy equivalence, then φ_* is an isom on π_n for all n.

<u>Cor</u>: If X is contractible then $\pi_n X = 0$ for all n.

<u>Prop</u>: A covering map $p : (\tilde{X}, \tilde{x}_0) \longrightarrow (X, x_0)$ induces an $\cong$ on π_n for all $n \geq 2$.

<u>Contrast</u>: H_* for $\mathbb{R}^2 \longrightarrow$ 

<u>Cor</u>: $\pi_n(T^k) = 0$ for all $n \geq 2$.

<u>Proof</u>: Consider $p_* : \pi_n(\tilde{X}, \tilde{x}_0) \longrightarrow \pi_n(X, x_0)$

<u>Onto</u>: As $\pi_1 S^n = 1$, we have a lift $\tilde{f}$ as shown and so

$$p_*([\tilde{f}]) = [f].$$

$$\begin{tikzcd}
 & (\tilde{X}, \tilde{x}_0) \arrow[d, "p"] \\
(S^n, s_0) \arrow[ur, dashed, "\tilde{f}"] \arrow[r, "f"] & (X, x_0)
\end{tikzcd}$$

<u>1-1</u>: Suppose $g \in \pi_n(\tilde{X}, \tilde{x}_0)$ goes to 0 under p_*.

That is $g \circ p = 0$ in $\pi_n(X, x_0)$, i.e. $g \circ p \simeq$ const map.

By the covering homotopy property, g is homotopic to a lift of Const_{x_0} which must be $\mathrm{Const}_{\tilde{x}_0}$.

That is $g = 0$ in $\pi_n(\tilde{X}, \tilde{x}_0)$