Lecture 25: Proof of excision

Thm: X a CW complex which is the union of subcomplexes A and B with $C = A \cap B \neq \emptyset$. If (A,C) is m-connected and (B,C) is n connected, then $\pi_i(A,C) \to \pi_i(X,B)$ is an isom for $i < n+m$ and onto for $i = n+m$.

Cor: (X,A) an r-conn CW pair, A is s-conn.
Then $\pi_i(X,A) \to \pi_i(X/A)$ is an $\cong$ for $i \leq r+s$
onto for $i = r+s+1$

Pf: $X \cup CA \simeq_{h.e.} X \cup CA / CA = X/A$

Apply excision to $(X,A) \longrightarrow (X \cup CA, CA)$. Extra dimension comes from (CA, A) being $s+1$ connected. ∎

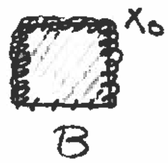
Pf of Excision: [Will start with some simple cases.]

Note if $A \setminus C$ has no cells of dim $\leq n$ then (A,C) is n-connected. [Cellular approx, can event. reduce to this.]

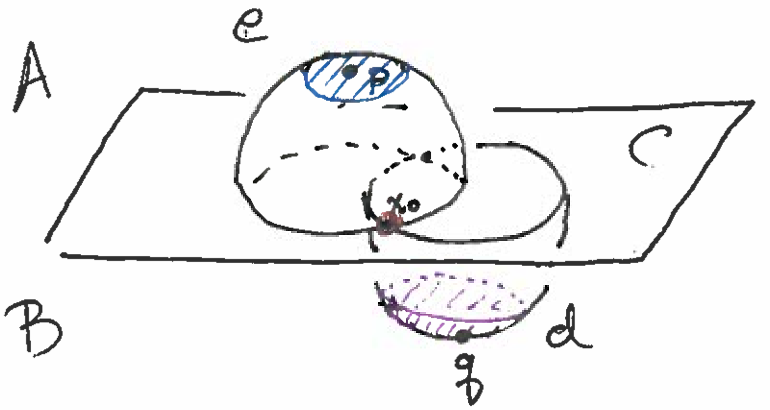
Simplest Case. $A = C \cup e^2 \qquad B = C \cup d^2$

Everything, including (X,B) is 1-connected.

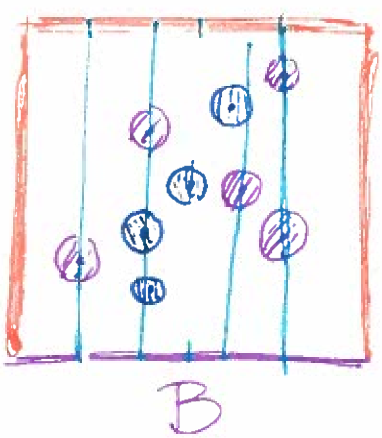
Q: Why is $\pi_2(A,C) \longrightarrow \pi_2(X,C)$ onto?

Consider: f:  $\longrightarrow X$ Pick $p \in e$ and $q \in d$.

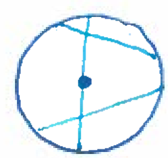
Homotope f to be "nice", that is $f^{-1}(p), f^{-1}(q) =$ finite number of points and f locally a linear homeom near each such pt.



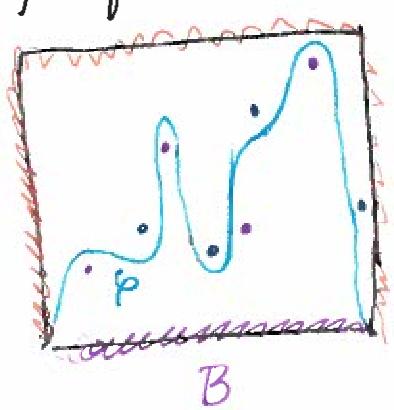
Claim: Can choose p so that no pt of $f^{-1}(p)$ is one the same vertical line as a pt in $f^{-1}(q)$



Reason: Image of lines in e has to low a dimsion

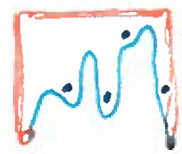


Now can chose $\varphi: I \to I$ so all $f^{-1}(q)$ is below the graph and all $f^{-1}(p)$ is above it and $\varphi = 0$ on ∂I.



Consider

$$\begin{tikzcd}
\pi_2(A, C) \arrow[r] \arrow[d, "\cong"'] \arrow[dr, phantom, "\circlearrowright"] & \pi_2(X, B) \arrow[d, "\cong \text{ as } X \setminus p \text{ def retracts to } B"] \\
\pi_2(X \setminus q, X \setminus \{p, q\}) \arrow[r] & \pi_2(X, X \setminus p)
\end{tikzcd}$$

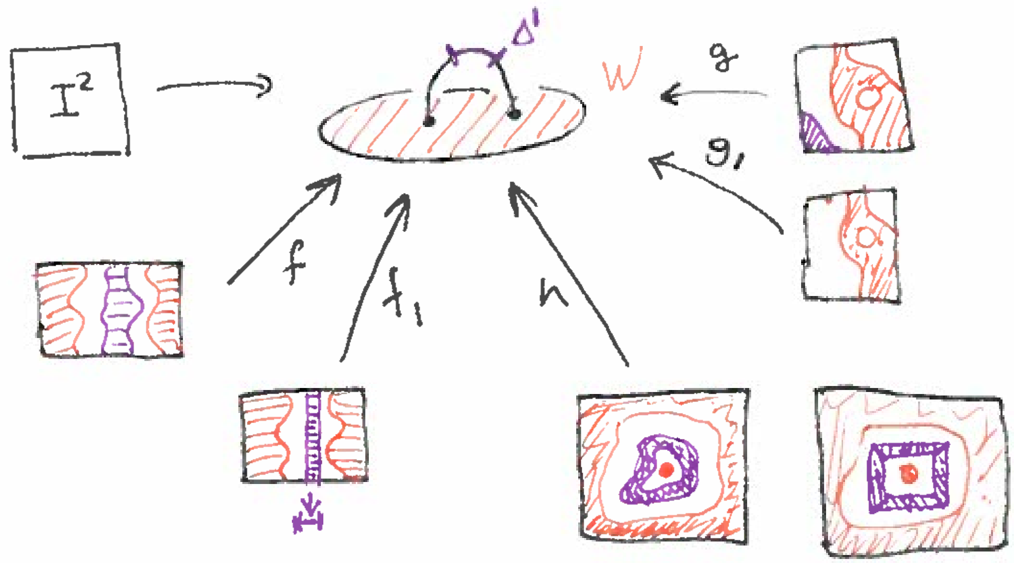
In $\pi_2(X, X \setminus p)$, f is homotopic to 

which is in $\pi_2(X \setminus q, X \setminus \{p,q\})$. Thus the horizontal maps are surjective. //

Now suppose e has dim $m+1$ and f has dim $n+1$.

<u>Lemma 4.10</u>: Let $f : I^n \to Z = W \cup e^k$. Then f is homotopic, rel $f^{-1}(W)$ to a map f_1 such that $\exists$ a simplex $\Delta^k \subseteq e^k$ where $f_1^{-1}(\Delta^k)$ is a union of finitely many convex polyhedra on each of which f_1 is the restriction of a projection $\mathbb{R}^n \to \mathbb{R}^k$.

Examples:



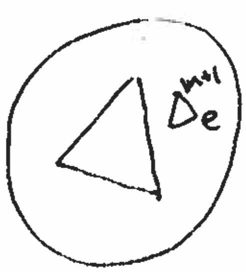
Do this for $f \in \pi_i(X, B)$ w.r.t. simplices $\Delta_e^{m+1} \subseteq e$

$\Delta_d^{n+1} \subseteq d$

Thus $f^{-1}(p)$ is a finite union of polyhedra of dim $i - m - 1$

$f^{-1}(q)$ is a finite union of polyhedra of dim $i - n - 1$

Idea: Choose p so that no pt of $f^{-1}(p)$ and $f^{-1}(q)$ are on the same vertical line. Since



e^{m+1}

$f\left(\{(s_1, s_2, \dots, s_{i-1}, t) \mid (s_1, \dots, s_i) \in f^{-1}(q),\ t \in I\}\right)$

intersect Δ_e^{m+1} is a polyhedron of dim $i - n$, can do provided $i - n < m + 1$. Now do same argument with φ as before. Hence crucial condition is $i \le n + m$. 1-1 is similar but with homotopies you lose one dimension.

Next consider case where $A = C \cup \underset{m+1 \text{ cells}}{\text{many}}$

$B = C \cup \underset{n+1 \text{ cells}}{\text{many}}$

When A or B has cells above dim $m+1 / n+1$,

induct on skeleta.

Cor Prop 4.13: X is homotopy equiv to

X', fixing C, such that $A = C \cup$ cells of dim $> m$

$B = C \cup$ cells of dim $> n$