Lecture 27: Eilenberg-MacLane spaces

A $K(G,n)$ is a CW complex X where $\pi_n X = G$ and all other $\pi_i X = 0$.

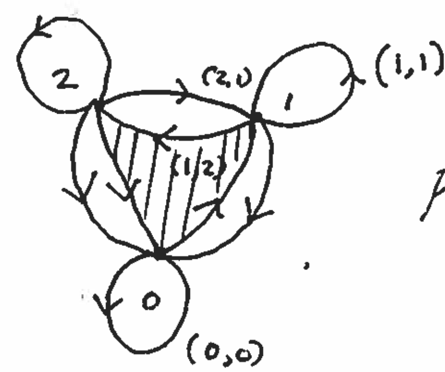
Thm: Any group G has a $K(G,1)$. If G is abelian, then $\exists\, K(G,n)$ for all n. These spaces are unique up to homotopy equivalence.

Pf of existence: $\underline{n=1}$. [Need to find X with $\pi_1 = G$ and contractible univ. cover.]

EG = Δ-complex with one n-simplex for each ordered $n+1$ tuple $(g_0, \dots, g_n)$ of elts of G.

[Vertices $\leftrightarrow G$ and simplices are glued by obv. restriction rel.]

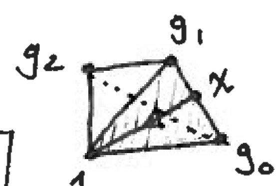
Ex: $G = \mathbb{Z}/3\mathbb{Z}$



plus $\underline{\underline{26}}$ more 2-simplices.

Contractible: Given $x \in \Delta_{(g_0, \dots, g_n)}$ homotope to 1_G via the straight line in $\Delta_{(1, g_0, \dots, g_n)}$

[Doesn't matter which simplex we regard x as in:]



G acts freely on EG by $\Delta_{(g_0,\dots,g_n)} \xrightarrow{h} \Delta_{(hg_0, hg_1, \dots, hg_n)}$

linear map.

Set $BG = EG/G$ a Δ-complex. Then $EG \to BG$

is a covering map and BG is a $K(G,1)$.

$\underline{n>1}$: <u>Lemma</u>: $\bigvee_\alpha S^n$ has $\pi_i = 0 \quad i<n$, $\pi_n = \bigoplus_\alpha \mathbb{Z}$ [Pf of this below.]

Consider $0 \to K \xrightarrow{i} \bigoplus_\alpha \mathbb{Z} \to G \to 0$ ← ident with $\bigvee_\alpha S^n$

Attach an $n+1$ cell to $\bigvee_\alpha S^n$ for each $\beta \in K$ via

$\partial D^{n+1} \xrightarrow[\varphi_\beta]{} \bigvee_\alpha S^n$ with $\varphi_\beta = i(\beta)$ to get a CW complex X.

$$\pi_{n+1}(X, \vee_\alpha S^n_\alpha) \xrightarrow{\partial} \pi_n(\vee_\alpha S^n_\alpha) \longrightarrow \pi_n(X) \longrightarrow \pi_n(X, \vee_\alpha S^n_\alpha) = 0$$

$\pi_{n+1}(X, \vee_\alpha S^n_\alpha) \cong \pi_{n+1}(X/\vee_\alpha S^n_\alpha) = \pi_{n+1}(\bigvee_\beta S^{n+1}_\beta) = \bigoplus_\beta \mathbb{Z}$

as $\vee_\alpha S^n_\alpha$ is $n-1$ conn. and $(X, \vee_\alpha S^n_\alpha)$ is n-conn.

Since ∂ is exactly i, get

$$\pi_n(X) \cong G$$

So have X which is $n-1$ connected and $\pi_n = G$

But $\pi_{n+1} X$ might be non-zero. If so, attach a bunch of $n+2$ cells along a gen set for $\pi_{n+1} X$ to get X_2. Note $\pi_i X_2 = \pi_i X$ for $i \le n$ since $X_2^{(n+1)} = X$. Lather, rinse, repeat, to get a $K(G, n)$. ∎

<u>Pf of Lemma:</u> Suppose there are finitely many α.

Then $$\bigvee_\alpha S^n_\alpha \hookrightarrow \prod_\alpha S^n_\alpha$$

as the n-skeleton, and all other cells of $\prod_\alpha S^n_\alpha$ have dim $\geq 2n$. So by cellular approx,

$$\pi_n(\vee_\alpha S^n_\alpha) \cong \pi_n(\textstyle\prod_\alpha S^n_\alpha) = \bigoplus_\alpha \mathbb{Z}.$$

[When there are infinitely many α, the topological product $\prod_\alpha S^n_\alpha$ is not obviously a CW complex.]

Consider $\bigoplus_\alpha \mathbb{Z} \xrightarrow{\Phi} \pi_n \vee_\alpha S^n_\alpha$ where

$$\mathbb{Z}_\alpha \longrightarrow \left(\text{image of } \pi_n S_\alpha \longrightarrow \pi_n \vee_\alpha S^n_\alpha\right)$$

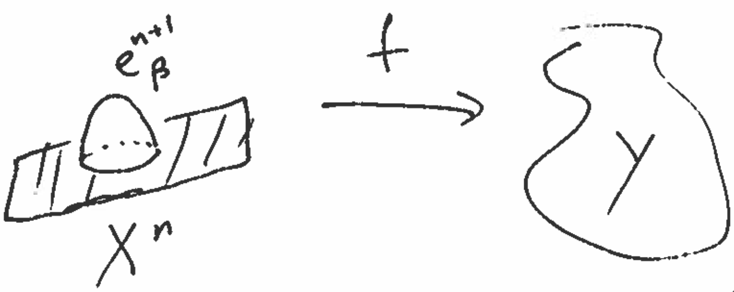
Φ is surjective since any $f : S^n \to \bigvee_\alpha S^n_\alpha$ has cpt image hence contained in some finite wedge of S^n_α.

Φ is injective since any null homotopy is again contained in some finite wedge. ▨

[Q: What prevents us from doing the 2nd construction when $n=1$? Actually nothing!]

<u>Pf of uniqueness</u>: Enough to show that any $K(G,n)$ Y is homotopy equivalent to the one X constructed above. Let $\psi : \pi_n X \to \pi_n Y$ be an isom.

Define $f : X^{(n)} \to Y$ via $S^n_\alpha \xrightarrow[\psi([S^n_\alpha])]{} Y$, so that f "implements" ψ. Extends over any $n+1$ cell e^{n+1}_β by noting that $f_*(\partial e^{n+1}_\beta) = \psi(\partial e^{n+1}_\beta) = 0$ and hence $f|_{\partial e^{n+1}_\beta}$ extends over e^{n+1}_β.



Repeat and then apply Whitehead's Thm. ▨