Lecture 28: Homology and homotopy.

Thm. For any G, there is a $K(G,1)$. For abelian G, there is a $K(G,n)$ for all $n \geq 1$. These spaces are unique up to homotopy equivalence.

Lemma: $X = \bigvee_\alpha S^n_\alpha$ has $\pi_i = 0$ for $i < n$ and $\pi_n = \bigoplus_\alpha \mathbb{Z}$

$\boxed{n \geq 2}$

Pf: If there finitely many α, then X is the n-skelton of $\prod_\alpha S^n_\alpha$. Since all other cells of $\prod_\alpha S^n_\alpha$ have dim $\geq 2n$, get

$$\pi_i X \cong \pi_i \prod_\alpha S^n_\alpha \cong \prod_\alpha \pi_i S^n_\alpha \text{ for } i \leq n.$$

[When ∞-many spheres $\prod_\alpha S^n_\alpha$ is not obviously a CW complex.]

Consider $\bigoplus_\alpha \mathbb{Z} \xrightarrow{\Phi} \pi_n X$ where $\mathbb{Z}_\alpha$ goes to the image of $\pi_n S^n_\alpha \xrightarrow{i_*} \pi_n X$. Φ is onto since any $f: S^n \to X$ has image contained in a finite subwedge; same for homotopies, so Φ is 1-1. ▨

For any space X have homomorphisms

$$h : \pi_k X \to H_k(X;\mathbb{Z})$$
$$(f : S^k \to X) \longmapsto f_*([S^k])$$

but often these aren't useful: S^n has little H_* much π_*
$\mathbb{C}P^\infty$ lots of H_*, little π_*

<u>Hurewicz Thm</u>: If Y is an $(n-1)$ connected CW complex where $n \geq 2$, then $\tilde{H}_i(Y) = 0$ for $i < n$ and $\pi_n Y \cong H_n(Y)$ via h.

Very useful in computing π_2 by passing to the univ. cover

$$\pi_2 Y \cong \pi_2 \tilde{Y} \cong H_2(\tilde{Y}).$$

$$\begin{array}{cl} \tilde{Y} & \pi_1 = 1 \\ \downarrow & \\ Y & \end{array}$$

<u>Pf</u>: By Cor 4.16, can assume that Y has a single 0 cell and no other cells of dim $< n$. Hence by cellular homology see that $\tilde{H}_i(Y) = 0$ for $i < n$.

As both H_n and π_n are det by $Y^{(n+1)}$, can assume Y has no cells of dim $> n+1$. So

$$Y = (\underbrace{\bigvee_\alpha S^n_\alpha}_{X}) \cup_\beta e^{n+1}_\beta$$

As with the $K(G,n)$ theorem, we have

$$\begin{tikzcd}
\pi_{n+1}(Y,X) \arrow[r, "\partial"] \arrow[d, "\text{obvious isom or relative Hurewicz map}"'] & \pi_n(X) \arrow[r] \arrow[d, "h \ \cong \text{ pretty clearly}"] & \pi_n(Y) \arrow[r] \arrow[d, "h"] & \pi_n(Y,X) \arrow[d, "\cong"] \\
H_{n+1}(Y,X) \arrow[r] & H_n(X) \arrow[r] & H_n(Y) \arrow[r] & H_n(X,Y)
\end{tikzcd}$$

$\pi_{n+1}(Y,X) \cong \pi_{n+1}(\bigvee S^{n+1}_\beta) \cong \bigoplus_\beta \mathbb{Z}$, $\quad \pi_n(X) = \bigoplus_\alpha \mathbb{Z}$, $\quad \pi_n(Y,X) = 0$

$H_{n+1}(Y,X) \cong \bigoplus_\beta \mathbb{Z}$, $\quad H_n(X) = \bigoplus_\alpha \mathbb{Z}$, $\quad H_n(X,Y) = 0$

<u>Key</u>: $\pi_{n+1}(Y,X)$ is gen by $D^{n+1} \xrightarrow{f_\beta} e^{n+1}_\beta$

which has ∂ given by $S^n \xrightarrow[\phi_\beta]{} X$ where ϕ_β is

the attaching map for e^{n+1}_β.

The components of $\partial f_\beta = \phi_\beta \in \pi_n X = \bigoplus_\alpha \mathbb{Z}$

can be computed by the compositions

$$S^n \xrightarrow{\phi_\beta} X \xrightarrow{p_\alpha} X \Big/ \bigvee_{\alpha' \neq \alpha} S^n_{\alpha'} = S^n_\alpha$$

That is

$$\partial f_\beta = \sum \underbrace{[\, p_\alpha \circ \phi_\beta \,]}_{\in \pi_n S^n_\alpha}$$

and $[p_\alpha \circ \phi_\beta] \in \mathbb{Z}$ is just the degree of $p_\alpha \circ \phi_\beta$.

But this matches the cellular boundary formula

$$\partial e^{n+1}_\beta = \sum_\alpha d_{\alpha\beta} \cdot S^n_\alpha$$

degree of $e^{n+1}_\beta \longrightarrow \bigvee_\alpha S^n_\alpha \Big/ \bigvee_{\alpha' \neq \alpha} S^n_{\alpha'}$

and so left hand square commutes.

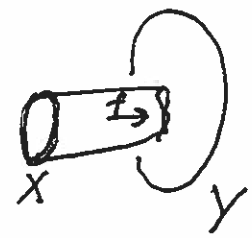
Thus $\pi_n Y \xrightarrow{h} H_n Y$ is an isomorphism

by the 5-lemma. $\blacksquare$

<u>Relative Hurewicz</u>: (X,A) a CW pair which is $(n-1)$ connected for $n \geq 2$ with A simply connected and nonempty. Then $H_i(X,A) = 0$ for $i < n$ and $\pi_n(X,A) \cong H_n(X,A)$.

<u>Pf</u>: By excision, $\pi_i(X,A) \cong \pi_i(X/A)$ for $i \leq (n-1)+1 = n$. Of course $H_i(X,A) \cong \tilde{H}_i(X/A)$ for all i. Now apply original Hurewicz to X/A which is $n-1$ connected. $\blacksquare$

Cor: A map $f: X \to Y$ between simply connected CW complexes is a homotopy equivalence iff $f_*: H_n(X) \to H_n(Y)$ is an isomorphism for all n.

Pf: Replacing Y by the mapping cylinder 

can assume f is the inclusion of a subcomplex. Since X and Y are simply connected, we have that $\pi_1(Y, X) = 0$. The relative Hurewicz Thm says that the first non-zero $\pi_n(Y, X)$ is isomorphic to the first non-zero $H_n(Y, X)$.

But all the $H_n(Y, X) = 0$. Hence all $\pi_n(Y, X) = 0$

$\Rightarrow$ all $\pi_n X \to \pi_n Y$ are $\cong$ $\Longrightarrow$

$X \simeq_{h.e.} Y$ by Whitehead.