# Lecture 18: Second proof of Poincaré Duality.



Previously on Math 526: $M^n$ an $R$-oriented mfld.

For $K^{cpt} \subseteq M$ define $D_K : H^k(M|K) \to H_{n-k}(M)$
$$\varphi \longmapsto \mu_K \cap \varphi$$
where $\mu_K \in H_n(M|K)$ is the unique elt with $\mu_K \mapsto$ chosen orient at $x \in K$ in $H_n(M|x)$

Set $D_M : H^k_c(M) \longrightarrow H_{n-k}(M)$ to be the induced map on $\varinjlim_{K^{cpt} \subseteq M} H^k(M|K)$, where $H^k_c(M) \cong \varinjlim_{K^{cpt} \subseteq M} H^k(M|K)$.

Thm: $M^n$ an $R$-oriented mfld. Then $D_M$ is an $\cong$ for all $k$.

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Inductive steps:

(A) $U, V \underset{\text{open}}{\subseteq} M$. If $D_U$, $D_V$ and $D_{U \cap V}$ are $\cong$, so is $D_{U \cup V}$.

(B) Suppose $M$ is the union of $U_1 \subseteq U_2 \subseteq U_3 \subseteq \dots$ where each $D_{U_i}$ is an isom. Then $D_M$ is an isom.

Base cases: ① $D_{\mathbb{R}^n}$ is an isom.

② If $U \underset{\text{open}}{\subseteq} \mathbb{R}^n$, then $D_U$ is an isom.

Pf of Thm: Any $M$ is a countable union of open $V_j$ with $V_j \cong$ open in $\mathbb{R}^n$. Set $U_i = \bigcup_{j=1}^{i} V_j$. By ① and ② have $D_{V_j}$ and $D_{V_j \cap V_\ell}$ are $\cong$. Inductively [on all such open sets in $M$] get by Ⓐ that $D_{U_i}$ is an $\cong$. Done by Ⓑ. ▨

Pf of Ⓑ:

$$H_c^k(U_i) \overset{\text{last time}}{\cong} \varinjlim_{K \overset{cpt}{\subseteq} U_i} H^k(U_i \mid K) \overset{\text{excision}}{\cong} \varinjlim_{K^{cpt} \subseteq U_i} H^k(M \mid K)$$

$$\downarrow$$

Get map $H_c^k(U_{i+1}) \cong \varinjlim_{K^{cpt} \subseteq U_{i+1}} H^k(M \mid K)$

since second limit is over a larger collection of sets.

Now $\varinjlim_i H_c^k(U_i) \cong \varinjlim_{K \overset{cpt}{\subseteq} M} H^k(M \mid K) \cong H_c^k(M)$

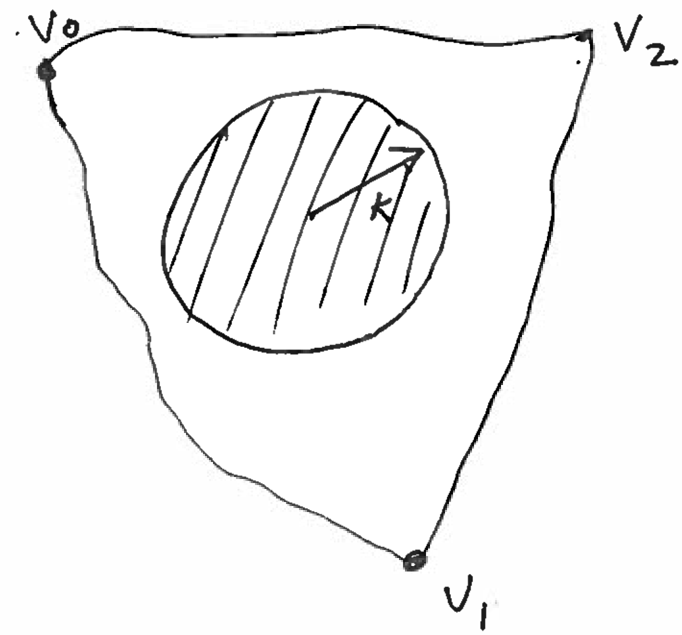
Have isom $D_{U_i} : H_c^k(U_i) \to H_{n-k}(U_i)$, giving

$$\underset{\longrightarrow}{\lim}\, D_{U_i} : H_c^k(M) \xrightarrow[\phantom{x}]{\cong\ \leftarrow \text{since each } D_{U_i} \text{ is.}} \underset{\longrightarrow}{\lim}\, H_{n-k}(U_i) \cong H_{n-k}(M)$$

$\parallel$

$D_M$. Thus $D_M$ is an isomorphism. ▨

① Need to calculate $D_{\mathbb{R}^n} : H_c^n(\mathbb{R}^n) \to H_0(\mathbb{R}^n)$




Consider a gen $\varphi$ of $H^n(\mathbb{R}^n \mid \overline{B}_k(0))$ and chain $\sigma : \Delta_n \to \mathbb{R}^n$ rep $\mu_k \in H_n(\mathbb{R}^n \mid \overline{B}_k(0))$

Then $\varphi(\sigma) = 1$ (say) and

$$\mu_k \cap \varphi = [v_n]$$

which generates $H_0(\mathbb{R}^n)$.

② Any $U \subseteq \mathbb{R}^n$ is a countable union of convex open sets $V_j$ (e.g. open balls). Set $U_i = \bigcup_{j=1}^{i} V_j$.

Inductively [on unions of $i-1$ convex open sets in $\mathbb{R}^n$] can use Ⓐ to show $D_{U_i}$ is an $\cong$ and Ⓑ to show $D_U$ is.

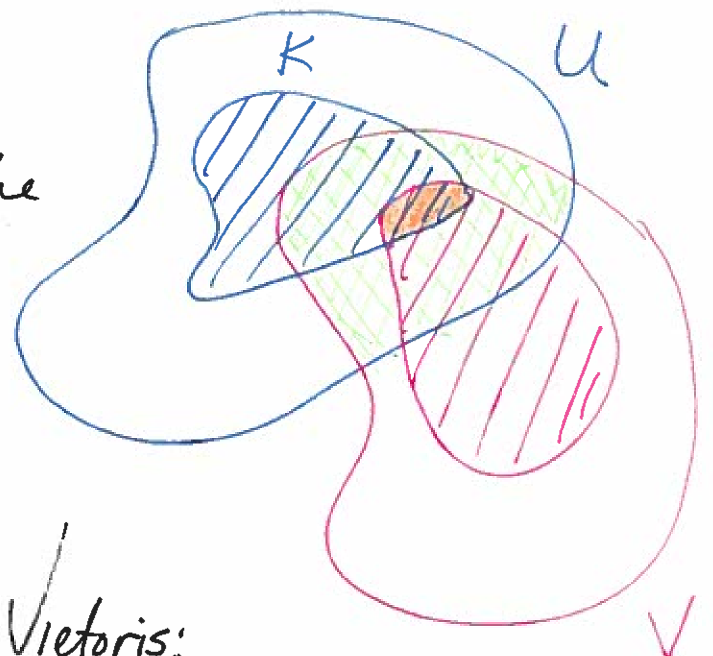
Ⓐ The following diagram commutes up to sign: 

$$\begin{tikzcd}
\to H_c^k(U\cap V) \arrow[r] \arrow[d, "D_{U\cap V}"'] & H_c^k(U)\oplus H_c^k(V) \arrow[r] \arrow[d, "D_U\oplus -D_V"] & H_c^k(U\cup V) \arrow[r] \arrow[d, "D_{U\cup V}"] & H_c^{k+1}(U\cap V) \arrow[r] \arrow[d, "D_{U\cap V}"] & \cdots \\
\to H_{n-k}(U\cap V) \arrow[r] & H_{n-k}(U)\oplus H_{n-k}(V) \arrow[r] & H_{n-k}(U\cup V) \arrow[r] & H_{n-k-1}(U\cap V) \arrow[r] & {}
\end{tikzcd}$$

[Query what's odd about the top row? Arrows backwards!]

Top row follows from if $K\subseteq U$, $L\subseteq V$ are cpt, then each

$$\to H^k(\underset{\text{or } M}{U\cap V}\mid K\cap L)\to H^k(\underset{\text{or } M}{U}\mid K)\oplus H^k(\overset{\text{or } M}{V}\mid L)\to H^k(\overset{\text{or } M}{U\cup V}\mid K\cup L)\to$$




is exact, and moreover the direct limit of exact is exact.

Bottom row is ordinary Mayer-Vietoris; vertical maps are all direct limits.

Assuming (almost) commutativity, get Ⓐ by the 5-lemma.

For proof of commutativity, see Lemma 3.36 in Hatcher.

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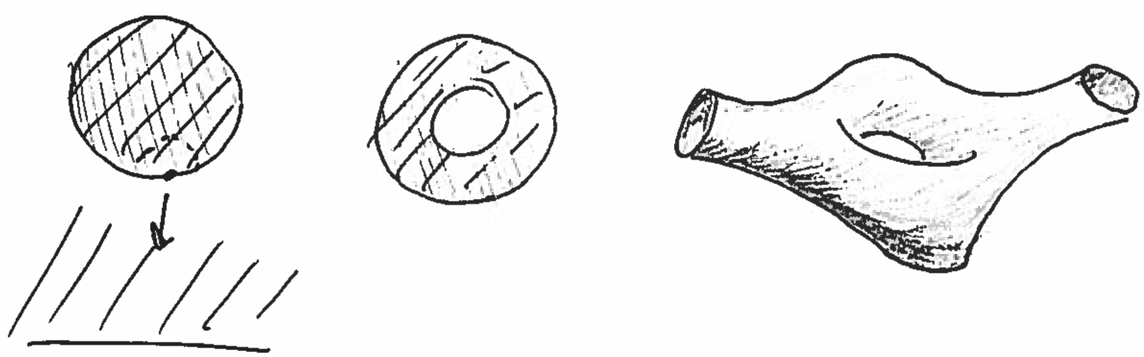
Other forms of duality.

Def: A Hausdorff $2^{nd}$ countable top space $M$ is an n-manifold w/ boundary if each $p \in M$ has a nbhd $U$ homeo to $\mathbb{R}^n$ or $\mathbb{R}^n_+ = \mathbb{R}^{n-1} \times [0,\infty)$



Thm Suppose $M^n$ is an $R$-orient. mfld w/ $\partial$. Then $H^k(M) \cong H_{n-k}(M, \partial M)$ and $H^k(M, \partial M) \cong H_{n-k}(M)$.

Also, Alexander Duality....