Lecture 9: Applications of cohomology

Last time: A, B are abelian gps

$$A \otimes B = \bigoplus_{(a,b) \in A \times B} \mathbb{Z}[a \otimes b] \Big/ \begin{array}{l} (a+a') \otimes b = a \otimes b + a' \otimes b \\ a \otimes (b+b') = a \otimes b + a \otimes b' \end{array}$$

Division Algebras: Bilinear $\mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$ with no zero divisors.

$$H^*(\mathbb{R}P^n; \mathbb{F}_2) = \mathbb{F}_2[\gamma] / (\gamma^{n+1}) \qquad |\gamma| = 1$$

$$H^*(\mathbb{R}P^n \times \mathbb{R}P^n; \mathbb{F}_2) = \mathbb{F}_2[\alpha, \beta] / (\alpha^{n+1}, \beta^{n+1})$$

$$\alpha = p_1^*(\gamma) \quad \beta = p_2^*(\gamma)$$

Correction: R ring $\quad A, B$ are R-modules

$$A \otimes_R B = \bigoplus_{(a,b) \in A \times B} R[a \otimes b] \Big/ \begin{array}{l} \text{same} \\ + (ra) \otimes b = a \otimes (rb) \end{array}$$

Also an R-module because of

Any abelian gp is a $\mathbb{Z}$-module

$$r \cdot a = \sum_{i=1}^{r} a$$

For ab gps A and B:

$$A \otimes B = A \otimes_{\mathbb{Z}} B$$

of groups $\qquad$ as $\mathbb{Z}$-modules.

Ex: $R \otimes_R R \cong R$ since $a \otimes b = ab \otimes 1$

So $\mathbb{Q} \otimes_{\mathbb{Q}} \mathbb{Q} \cong \mathbb{Q}$ but $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Q}$ is much larger
$\mathbb{C} \otimes_{\mathbb{C}} \mathbb{C} = \mathbb{C}$ but $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C} \cong \mathbb{R}^4$.

<u>Künneth Thm</u>: Suppose X and Y are spaces where $H^*(Y;R)$ is a finitely generated free R-module.

Then
$$H^*(X \times Y; R) \cong H^*(X;R) \otimes H^*(Y;R)$$

<u>Thm</u>: If $\mathbb{R}^d$ has the str of a division algebra, then $d = 2^k$.

<u>Pf</u>: Take $n = d-1$
Set $g : S^n \times S^n \longrightarrow S^n$ to be $g(x,y) = \dfrac{x \cdot y}{|x \cdot y|}$

[Makes sense because no 0-divisors.]

As $(-x)\cdot y = -(x \cdot y) = x \cdot (-y)$ we have
$$g(-x,y) = -g(x,y) = g(x,-y)$$

and so get a map $h : P^n \times P^n \longrightarrow P^n$ $\quad [P^n = \mathbb{R}P^n]$

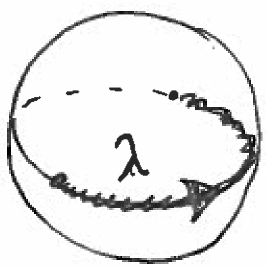
<u>Claim</u>: With $\mathbb{F}_2$ coeff's
$$\begin{array}{ccc} H^1(P^n \times P^n) & \xleftarrow{h^*} & H^1(P^n) \\ \alpha + \beta & \longleftarrow & \gamma \\ \overset{\shortparallel}{\gamma \times 1} \quad \overset{\shortparallel}{1 \times \gamma} & & \end{array}$$

[Note because of the cup product, this completely determines $H^*(P^n \times P^n) \leftarrow H^*(P)$.]

<u>Proof of Claim:</u> Take $n > 1$ so that $\pi_1 P^n = \mathbb{Z}/2\mathbb{Z}$.

Let's compute $\underset{\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}}{\pi_1(P^n \times P^n)} \xrightarrow{h_*} \underset{\mathbb{Z}/2\mathbb{Z}}{\pi_1(P^n)}$. Now

S^n



gives a gen of $\pi_1(P^n)$.

What is $h_*(1,0)$? Well, $(1,0) = (\lambda, \text{const}\, y_0)$

$\xmapsto{h_*} \lambda \cdot y_0 / |\lambda \cdot y_0|$. Basically, changing λ by the linear trans $\_ \cdot y_0$, so still get a path joining antipodal pts and hence $h_*(1,0) = 1$ and $h_*(0,1) = 1$

Same action on $H_1(\_ ; \mathbb{Z} \text{ or } \mathbb{F}_2)$ and since $H^1(\_ ; \mathbb{F}_2) = \mathrm{Hom}(H_1(\_ ; \mathbb{Z}), \mathbb{F}_2)$ we get the claim by dualizing. ∎

<u>Proof of Thm:</u> As $\gamma^d = 0$ in $H^*(P^n;\mathbb{F}_2)$ get

$$0 = h^*(\gamma^d) = (\alpha+\beta)^d = \sum_{k=0}^{d} \binom{d}{k} \alpha^k \beta^{d-k}$$

in $H^*(P^n \times P^n;\mathbb{F}_2)$. So $\binom{d}{k} \equiv 0 \mod 2$ for all $0 < k < d$. Equivalently in $\mathbb{F}_2[X]$ we have $(1+x)^d = 1 + x^d$. Write $d = d_1 + d_2 + \cdots + d_\ell$ where each d_i is a power of 2 and $d_1 < d_2 < \cdots < d_\ell$.

Then

$$\begin{aligned}(1+x)^d &= (1+x)^{d_1} \cdots\cdots (1+x)^{d_\ell} \\ &= (1+x^{d_1}) \cdots\cdots (1+x^{d_\ell}) \leftarrow \text{since } p \mapsto p^2 \text{ is an additive hom of } \mathbb{F}_2[x] \\ &= \text{some poly with } 2^\ell \text{ terms}\end{aligned}$$

$\Rightarrow \ell = 1$, i.e. $d = 2^\ell$. ▨

If time remains, talk about Poincaré Duality:

<u>Def</u>: An n-manifold is a Hausdorff, 2^{nd} countable, topological space where every pt has an open nbhd homeo to $\mathbb{R}^n$.

<u>Thm</u>: M compact connected n-mfld. $H_k(M;\mathbb{F}_2) \cong H_{n-k}(M;\mathbb{F}_2)$

If M is <u>orientable</u> then $H_k(M;\mathbb{Z}) \cong H^{n-k}(M;\mathbb{Z})$