Math 526, 9/5/14.

Cup Product As explained by NMD(?) (introduced?)

$$H^i(X) \times H^j(X) \xrightarrow{\cup} H^{i+j}(X).$$

Different (but equivalent) possible defs.

Most formal $H^i(X) \times H^j(X) \to H^{i+j}(X \times X) \xrightarrow{\Delta^*} H^{i+j}(X)$.

But more concretely:

R (comm?) ring, e.g. $\mathbb{Z}$, $\mathbb{Z}/n\mathbb{Z}$, $\mathbb{Q}$, etc. Define

$$C^k(X;R) \times C^l(X;R) \xrightarrow{\cup} C^{k+l}(X;R) \quad \Big(\text{cochain level}$$

$$\alpha \qquad \beta \longmapsto \alpha \cup \beta \quad)$$

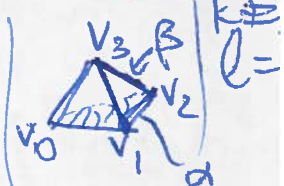
As follows: α is R-valued fn. on $\{\sigma_k : \Delta^k \to X\}$

β is R-valued fn on $\{\sigma_l : \Delta^l \to X\}$.

Given $\sigma : \Delta^{k+l} \to X$, can restrict to simplexes

$\{[v_0, \ldots, v_k, 0, \ldots, 0] \in \Delta^{k+l}\} \simeq \Delta^k$,

$\{[0, \ldots, 0, v_k, \ldots, v_{k+l}] \in \Delta^{k+l}\} \simeq \Delta^l$;



write, $\sigma|_{[v_0, \ldots, v_k]}$, $\sigma|_{[v_{k+1}, \ldots, v_{k+l}]}$ resp.

Then

Def $(\alpha \cup \beta)(\sigma) = \alpha(\sigma|_{[v_0, \ldots, v_k]}) \beta(\sigma|_{[v_k, \ldots, v_{k+l}]})$.

Picture! and extend linearly.

Lemma $\delta(\alpha \cup \beta) = \delta\alpha \cup \beta + (-1)^k \alpha \cup \delta(\beta)$

Pf. Calculate both sides. $\square$.

Corollary (1) If $\delta\alpha = 0$, $\delta\beta = 0$ then $\delta(\alpha \cup \beta) = 0$.

(2) If $\alpha = \delta\varphi$ and $\delta(\beta) = 0$ then

$$\alpha \cup \beta = \delta(\varphi) \cup \beta = \delta(\varphi \cup \beta).$$

(3). If $\delta\alpha = 0$ and $\delta(\varphi) = \beta$ then

$$\alpha \cup \beta = \alpha \cup \delta(\varphi) = (-1)^k \delta(\alpha \cup \varphi).$$

(4). $\cup$ induces a homomorphism.

$$H^k(X;R) \times H^l(X;R) \to H^{k+l}(X;R).$$

Rmk It is associative and distributive on chain level, hence on cohomology as well.

Warning $\alpha \cup \beta \neq \beta \cup \alpha$ in general.

[graded commutative on H^* however...].

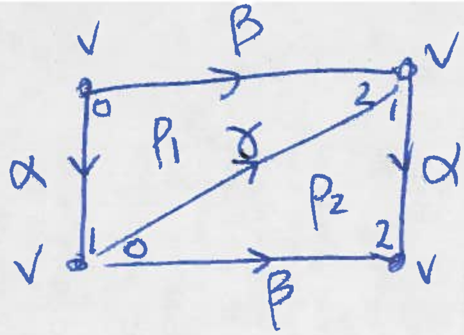
Set $H^*(X;R) = \bigoplus H^n(X;R)$.

Cor It's an R-alg under cup product, and an R-alg. w/ unit if R has a unit.

[$1_{H^*} \in H^0(X;R)$ def. by $1_{H^*}(0\text{-simplex}) = 1_R$.]

Ex 1

T^2



$$\dot{C}_* \qquad \begin{matrix}\mathbb{Z}\rho_1\\ \oplus \\ \mathbb{Z}\rho_2\end{matrix} \xrightarrow{\begin{pmatrix}1 & 1\\ -1 & -1\\ 1 & 1\end{pmatrix}} \begin{matrix}\mathbb{Z}\alpha\\ \oplus\\ \mathbb{Z}\beta\\ \oplus\\ \mathbb{Z}\gamma\end{matrix} \xrightarrow{0} \mathbb{Z}v$$

$$C^* \qquad \begin{matrix}\mathbb{Z}\rho_1^*\\ \oplus \\ \mathbb{Z}\rho_2^*\end{matrix} \xleftarrow{\begin{pmatrix}1 & -1 & 1\\ 1 & -1 & 1\end{pmatrix}} \begin{matrix}\mathbb{Z}\alpha^*\\ \oplus\\ \mathbb{Z}\beta^*\\ \oplus\\ \mathbb{Z}\gamma^*\end{matrix} \xleftarrow{0} \mathbb{Z}v^*.$$

Thus $H^0 \cong \mathbb{Z}$, $H^1 \cong \mathrm{span}\left\{\begin{pmatrix}1\\1\\0\end{pmatrix}, \begin{pmatrix}0\\1\\1\end{pmatrix}\right\} \cong \mathbb{Z}^2$,

$H^2 \cong \mathrm{coker}\begin{pmatrix}1 & -1 & 1\\ 1 & -1 & 1\end{pmatrix} \cong \mathbb{Z}^2/(1,1)\mathbb{Z} \cong \mathbb{Z}$ via

$(a,b) \longmapsto a-b$. (e.g.).

Now, can see from above:

- $\partial(\rho_1 - \rho_2) = 0$.
- Thus, if $\varphi \in C^1$ $(\delta\varphi)(\rho_1 - \rho_2) = 0$. Thus if, for some $\psi \in Z^2$, $\psi(\rho_1 - \rho_2) \neq 0$, then $[0] \neq [\psi]$ in H^2.
- Now compute: if $\psi_1, \psi_2 \in C^1$, then $(\psi_1 \cup \psi_2)(\rho_1) = \psi_1(\alpha)\psi_2(\gamma)$, $(\psi_1 \cup \psi_2)(\rho_2) = \psi_1(\gamma)\psi_2(\alpha$

Thus, consider
$$\left.\begin{aligned}\psi_1 &= (\alpha^* - \gamma^*) \\ \psi_2 &= (\alpha^* + \beta^*)\end{aligned}\right\} \in Z^1.$$

Now
$$\psi_1 \cup \psi_2 = \alpha^* \cup \alpha^* - \gamma^* \cup \alpha^* + \alpha^* \cup \beta^* - \gamma^* \cup \beta^*,$$
$$\psi_2 \cup \psi_1 = \alpha^* \cup \alpha^* - \alpha^* \cup \gamma^* + \beta^* \cup \alpha^* - \beta^* \cup \gamma^*.$$

Eval. on $\rho_1 - \rho_2$, ie. "pair with $\alpha \,..\, \gamma - \gamma \,..\, \alpha$", as explained above, get

$$(\psi_1 \cup \psi_2)(\rho_1 - \rho_2) = 1,$$
$$(\psi_2 \cup \psi_1)(\rho_1 - \rho_2) = -1.$$

So $[\psi_1] \cup [\psi_2] = -[\psi_2] \cup [\psi_1] \neq [0]$ in $H^2(T^2, \mathbb{Z}$

<u>Ex 2</u>

$S^1 \vee S^1 \vee S^2$:

Has same cohomology as a graded abelian group, but cup products on $H^1(S^1 \vee S^1 \vee S^2, \mathbb{Z})$ are always zero.

<u>Remark</u> When you study Poincaré duality, you'll relate cup products to ("geometric") intersections of cycles.

Some Additional Properties

Recall $C_n(X,A) = C_n(X)/C_n(A)$ for $A \subseteq X$ relative chains.

$$C^n(X,A;R) = \mathrm{Hom}(C_n(X,A),R).$$

$\ulcorner \varphi : C^n(X) \to R$ that vanish on ~~image~~ $C_n(A)$. $\lrcorner$

Get compatible maps

$$\begin{tikzcd}
H^k(X,A)\times H^l(X) \arrow[rr, bend left, "\smile"] & H^k(X,A)\times H^k(X,A) \arrow[l] \arrow[r, "\smile"] \arrow[d] & H^{k+l}(X,A) \\
& H^k(X)\times H^k(X,A) \arrow[ur, "\smile"] &
\end{tikzcd}$$

Functoriality $f : X \to Y$ continuous. Then

$f^* : H^*(Y) \to H^*(X)$ is a ring map, ie.

$$f^*(\alpha \cup \beta) = f^*(\alpha) \cup f^*(\beta).$$

Cross Product. $H^k(X)\times H^l(Y) \xrightarrow{\times} H^{k+l}(X\times Y)$

$$\alpha, \beta \longmapsto p_1^*(\alpha) \cup p_2^*(\beta).$$

Relative $H^k(X,A)\times H^l(Y,B) \xrightarrow{\times} H^{k+l}(X\times Y, A\times Y \cup X\times B)$