Lecture 20: A few last aspects of duality.

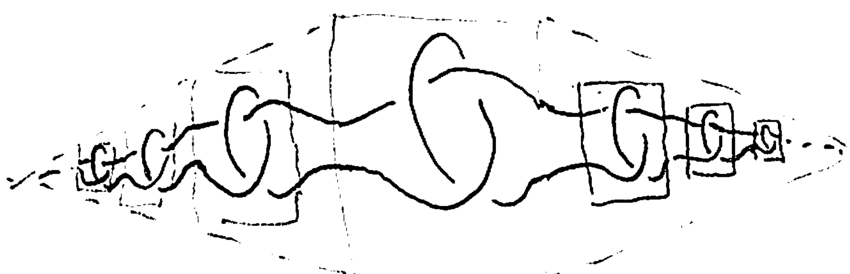
Alexander Duality: If $K \subseteq S^n$ is cpt, locally contractible, not $\emptyset$ or S^n, then for all i:

$$\tilde{H}_i(S^n \setminus K; \mathbb{Z}) \cong \tilde{H}^{n-i-1}(K; \mathbb{Z})$$

Cor: $\tilde{H}_*(S^n \setminus K)$ does not depend on how K is embedded in S^n.

$\pi_1(S^3 \setminus \bigcirc) = \mathbb{Z}$

Ex: $S^1 \hookrightarrow S^3$



$\Big\}$ $\pi_1(S^3 \setminus K)$ is infinitly gen

all have

$$\tilde{H}_i(S^3 \setminus S^1) = \tilde{H}^{2-i}(S^1) = \begin{cases} \mathbb{Z} & i = 1 \\ 0 & \text{otherwise.} \end{cases}$$

Ex: Wild spheres $S^2 \hookrightarrow S^3$. $S^3 \setminus S^2$ must have two comps but both can have ∞-gen π_1.

See Alexander horned sphere in text.

Proof sketch:

$$H_i(S^n \setminus K) \cong H_c^{n-i}(S^n \setminus K) \quad \text{[Poincaré]}$$

Q: What is the directed system?

$$\cong \varinjlim_{\substack{U \supseteq K \\ \text{open}}} \underbrace{H^{n-i}(S^n \setminus K \,|\, S^n \setminus U)}_{= H^{n-i}(S^n \setminus K,\, U \setminus K)}$$

$$\cong \varinjlim H^{n-i}(S^n, U) \underset{4}{\cong} \varinjlim \tilde{H}^{n-i-1}(U) \text{ if } i \neq 0.$$

$$\underset{5}{\cong} \tilde{H}^{n-i-1}(K)$$

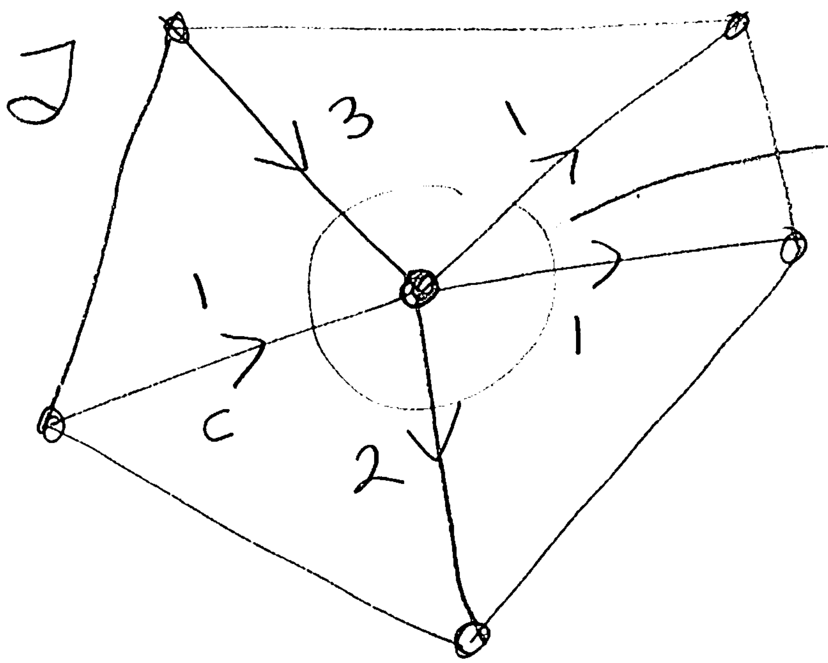
Step 4 is the long exact seq of (S^n, U)

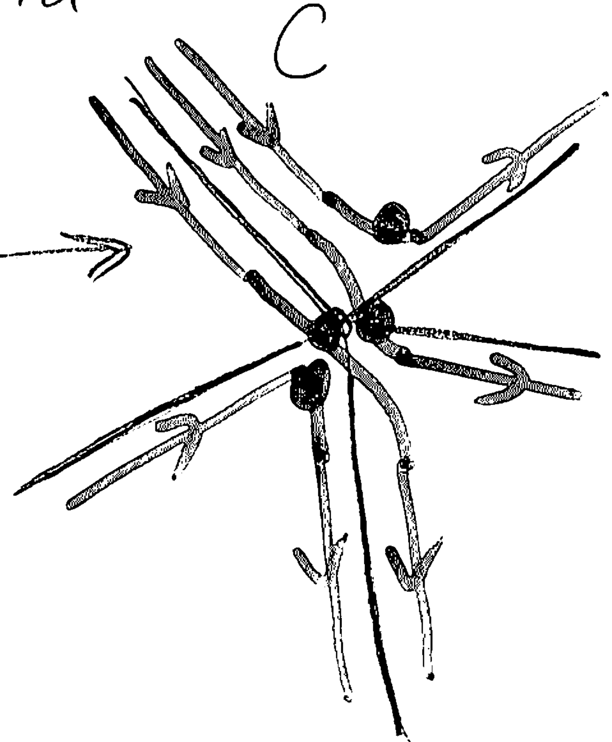
$$\leftarrow \underset{0 \;\, i\neq 0}{\tilde{H}^{n-i}(S^n)} \leftarrow \tilde{H}^{n-i}(S^n, U) \xleftarrow{\cong} \tilde{H}^{n-i-1}(U) \leftarrow \underset{0}{\tilde{H}^{n-i-1}(S^n)} \leftarrow$$

Step 5 is easy when K has an open nbhd U that def retracts to K. In general, only have retracts, so need to do a little point set topology. ▨

Poincaré Duality and 3-mflds:

<u>Warm up</u>: Every elt $[c] \in H_1(F^2; \mathbb{Z})$ can be rep by an emb. orient 1-mfld





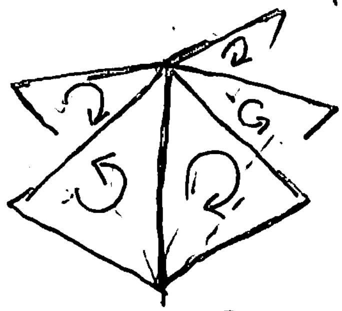
Get C an emb. orient triang. 1-mfld with $[C] = [c]$: The emb $C \hookrightarrow F$ is homotopic to a <u>cellular</u> map $C \xrightarrow{f} F$ where $f_{\#}(\text{fund cycle for } C) = c$ at the chain level.

<u>Prop</u>: Every $[c] \in H_2(M^3; \mathbb{Z})$ can be rep by an emb. surface.

<u>Sketch</u>: Let $\mathcal{J}$ be a triangl. of M^3 and c a cycle in $C_2(\mathcal{J})$.

Want to build surface from copies of the tris in $\mathcal{T}^{(2)}$ according to the weights det by c.

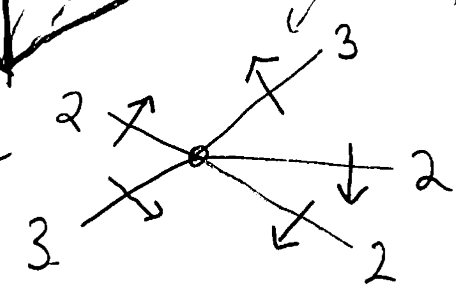
Will have several tris of $\mathcal{T}^{(2)}$ along each edge of $\mathcal{T}^{(1)}$



each orient by c.

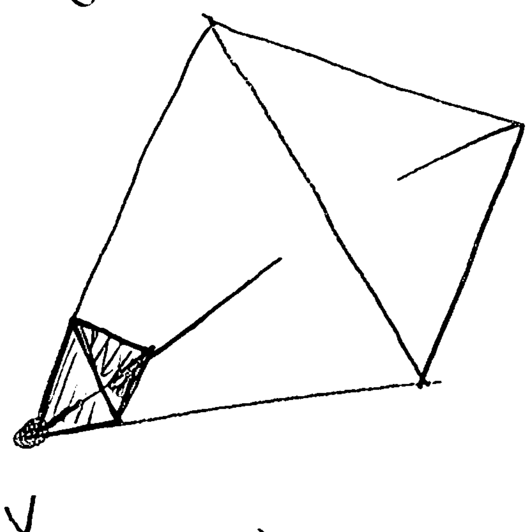
normal inducing orient shown

View from above



basically the 2D question along edges. But how about near $\mathcal{T}^{(0)}$?

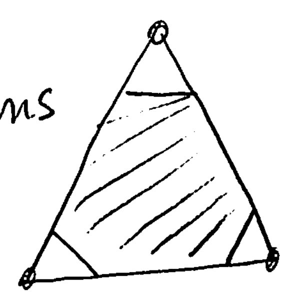
$N(v)$ made out of corners of the tets. $\partial N(v)$ is a surface. Must be S^2



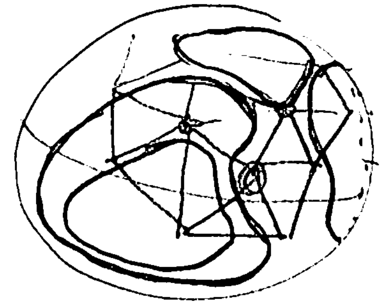
as $N(v) = C(\partial N(v))$

and we know $H_*(N(v), v) = H_*(M, v) = \tilde{H}_{*+1}(\partial N(v))$

First build most of S out of hexagons



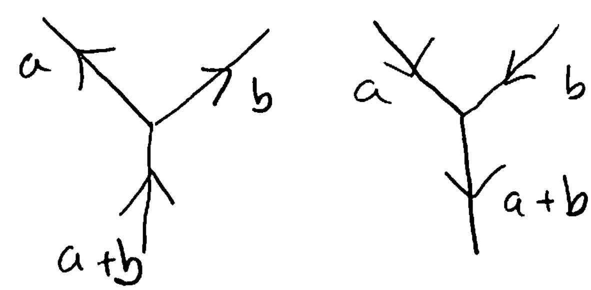
to get an emb. S in $M \setminus N(\mathcal{J}^{(0)})$. The ∂S meets each $\partial N(v)$ in circles



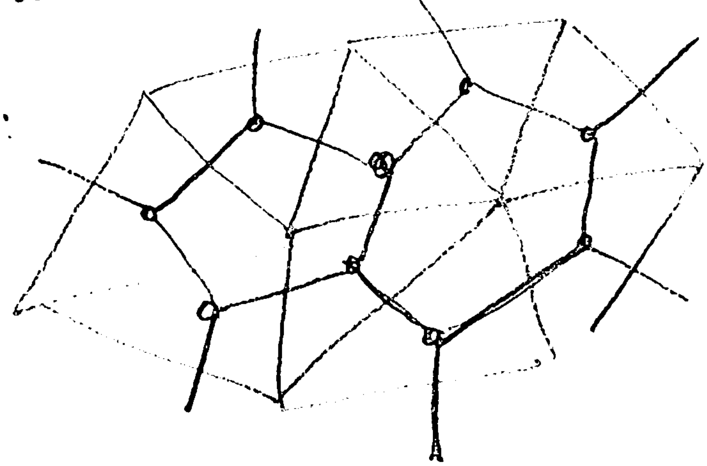
Now cap off each circle to get a clsd embed. $\bar{S}$ with a triang so $\bar{S} \hookrightarrow \mathcal{J}$ is hom. to a cellular map. f with $f_{\#}(\text{fund cycle } \bar{S}) = c$.

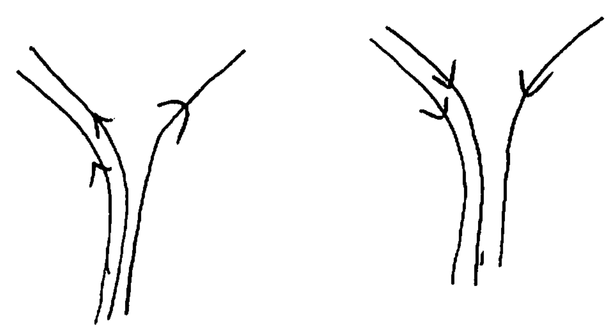
Or use the dual $\mathcal{D}$:

Only two poss (up to rot)

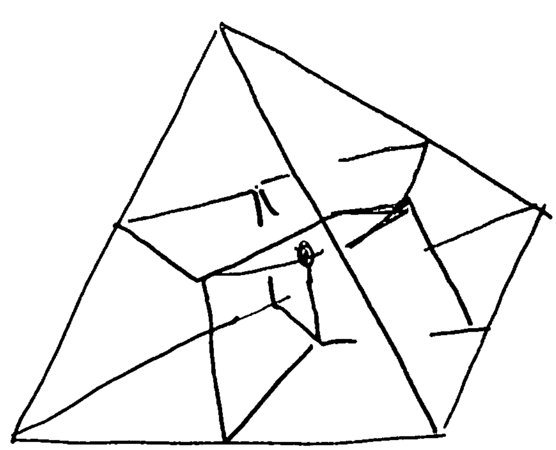


2D:



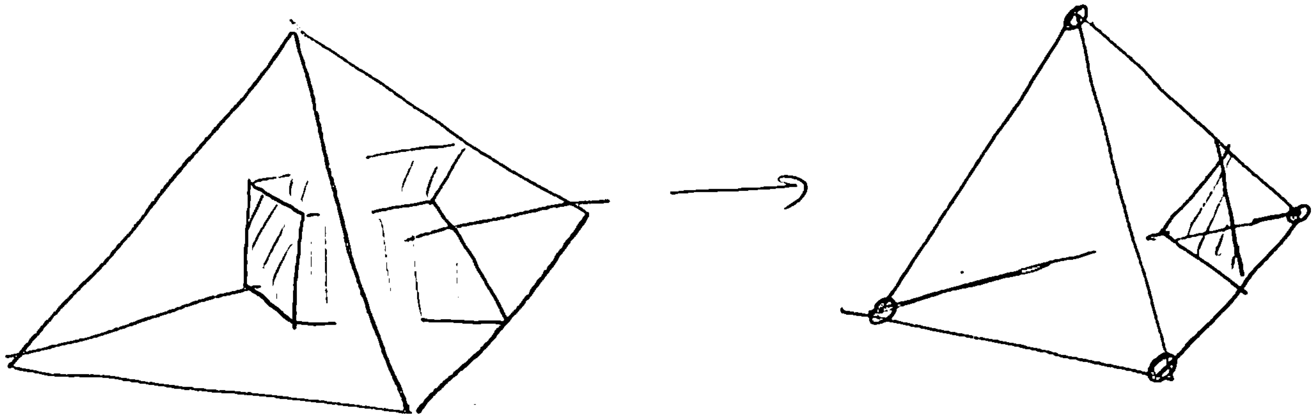


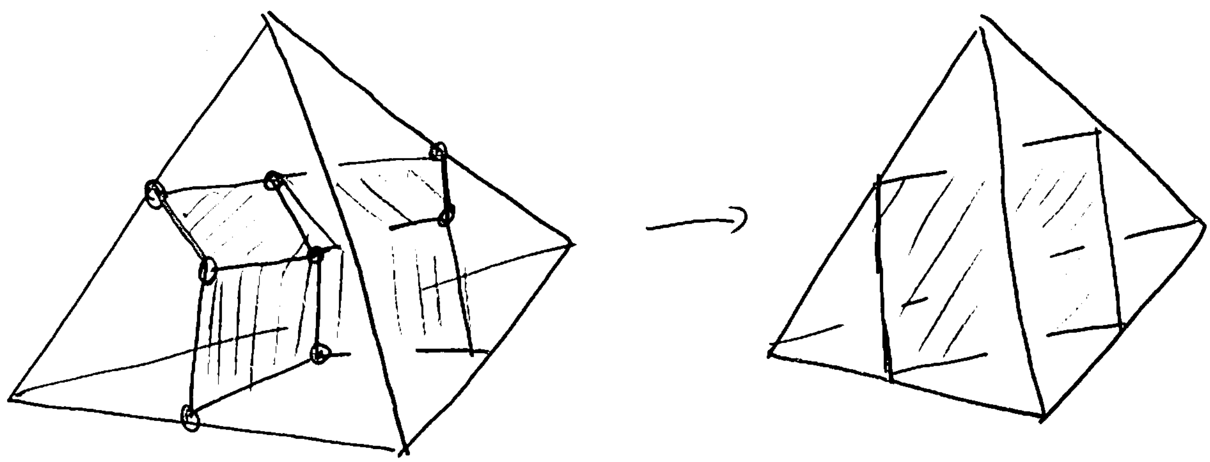
3D:



Use diff colors for bdry vs int. edges.

2 basic pieces:

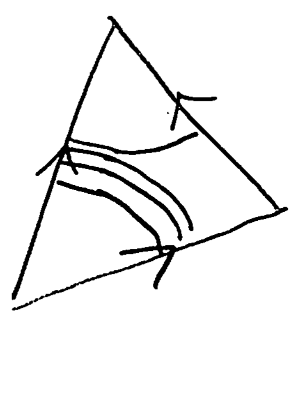




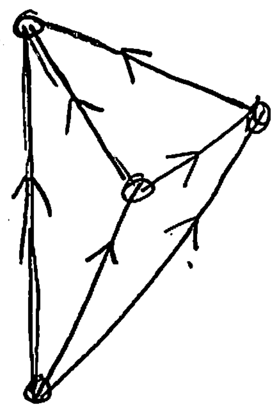
Including sym, there are 7 total.

Now show any weights on the dual 2-cells that give a cycle can be realized by a union of these pieces...

Hint: Each face looks like



Putting aside cases where some dual two cell has 0 weight, will always have a sink



so "pull off" a triangle piece near the sink...