Lecture 9: Applications of Cohomology

Division algebra: $\mathbb{R}^n \times \mathbb{R}^n \xrightarrow[\text{bilinear}]{} \mathbb{R}^n$ where $\forall a \neq 0, b \in \mathbb{R}^n$ both $ax = b$ and $ya = b$ are solvable.

Thm: When $\mathbb{R}^d$ is a division algebra $d = 2^k$.

Lemma: $H^*(\mathbb{R}P^n \times \mathbb{R}P^n; \mathbb{F}_2) \cong \mathbb{F}_2[\alpha, \beta] / (\alpha^{n+1}, \beta^{n+1})$

Here, if γ gen $H^1(\mathbb{R}P^n)$ then $\alpha = \gamma \times 1$ and $\beta = 1 \times \beta$.

Pf of Thm: Take $n = d-1$ and $P^n = \mathbb{R}P^n$. Set $g: S^n \times S^n \longrightarrow S^n$ to be $g(x,y) = \dfrac{x \cdot y}{\underbrace{|x \cdot y|}_{\text{[length as vec.]}}}$. [mult. in div. alg]

[Make sense because no 0-divisors.]

As $(-x) \cdot y = -(x \cdot y) = x \cdot (-y)$ we have

$$g(-x, y) = -g(x,y) = g(x, -y)$$

and so get a map $h: P^n \times P^n \longrightarrow P^n$.

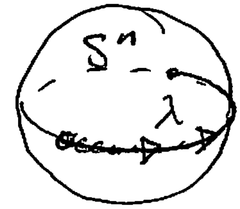
Claim: With $\mathbb{F}_2$ coeffs $H^1(P^n \times P^n) \xleftarrow{h^*} H^1(P^n)$

$$\alpha + \beta \longleftarrow\!\shortmid \gamma$$

[Because of the cup. product, this completely determines $H^*(P^n \times P^n) \longleftarrow H^*(P^n)$]

<u>Pf of Claim</u>: Take $n > 1$ so that $\pi_1 P^n = \mathbb{Z}/2\mathbb{Z}$.

Let's compute $\underset{\mathbb{Z}/2 \oplus \mathbb{Z}/2}{\pi_1(P^n \times P^n)} \xrightarrow{h_*} \underset{\mathbb{Z}/2}{\pi_1(P^n)}$. Now

a gen. of $\pi_1 P^n$ is the image of 

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What is $h_*(1,0)$? Well, $(1,0) = (\lambda, \text{const}_{y_0}) \overset{h_*}{\longmapsto}$ $\lambda \cdot y_0 / |\lambda \cdot y_0|$. Basically, changing λ by the linear trans $\_\!\_ \cdot y_0$, so still get a path joining antipodal pts. Hence $h_*(1,0) = h_*(0,1) = 1$.

By Hurewitz, get same action on $H_1(\_\,; \mathbb{Z}$ or $\mathbb{F}_2)$ and since $H^1(\_\,; \mathbb{F}_2) = \mathrm{Hom}(H_1(-;\mathbb{Z}), \mathbb{F}_2)$ we get the claim by dualizing. ▨

<u>Proof of Thm</u>: As $\gamma^d = 0$ in $H^*(P^n)$, get

$$0 = h^*(\gamma^d) = (\alpha + \beta)^d = \sum_{k=0}^{d} \binom{d}{k} \alpha^k \beta^{d-k}$$

in $H^*(P^n \times P^n)$ where we use $\mathbb{F}_2$ coeffs.

So $\binom{d}{k} \equiv 0 \mod 2$ for all $0 < k < d$.

Equiv, in $\mathbb{F}_2[x]$ have $(1+x)^d = 1 + x^d$. Write $d = d_1 + d_2 + \cdots + d_\ell$ where each d_i is a power of 2 and $d_1 < d_2 < \cdots < d_\ell$. Then

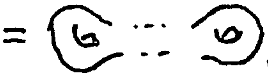
$$\begin{aligned}(1+x)^d &= (1+x)^{d_1} \cdot \cdots \cdot (1+x)^{d_\ell} \\ &= (1+x^{d_1}) \cdot \cdots \cdot (1+x^{d_\ell}) \\ &= \text{some poly with } 2^\ell \text{ nonzero terms.}\end{aligned}$$

since $p(x) \longmapsto p(x)^2$ is an additive hom of $\mathbb{F}_2[x]$.

Thus if $(1+x)^d = 1 + x^d$, have $\ell = 1$, i.e. $d = d_1 = 2^k$. ▨

[On to Poincaré duality...]

<u>Def</u>: An n-manifold is a Hausdorff, 2^{nd} countable, topological space where every point has an open nbhd homeo to $\mathbb{R}^n$.

<u>Ex</u>: $\mathbb{R}^n$, S^n, T^n, $\mathbb{RP}^n$, $\mathbb{CP}^n$, $L(p,q)$, $M_g =$  ...

[Geometric topology: study of such. For now, focus on H_* and H^*]

To do analysis, need <u>smooth</u> manifolds.

includes vector fields

Compact and w/o boundary

<u>Poincaré Duality</u>: M a closed connected n-mfld.

Then $H_k(M;\mathbb{F}_2) \cong H_{n-k}(M;\mathbb{F}_2) \cong H^{n-k}(M;\mathbb{F}_2)$.

If M is <u>orientable</u> then $H_k(M;\mathbb{Z}) \cong H^{n-k}(M;\mathbb{Z})$.

[Surprising since being a manifold is a purely local cond.]

<u>Ex</u>: <u>Cor</u>: For a clsd M^3, $H_1(M;\mathbb{F}_2) \neq 0 \iff$ $H_2(M;\mathbb{F}_2)$.

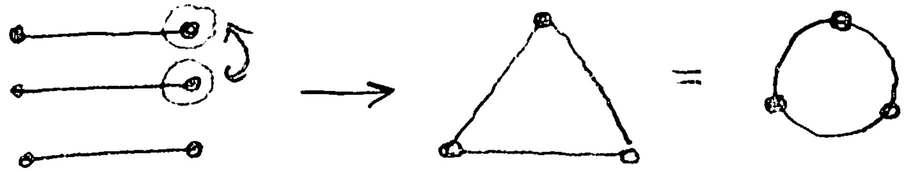
<u>Thm</u>: M closed conn. n-mfld. Then $H_n(M;\mathbb{Z})$ is either $\mathbb{Z}$ or 0 and $H_n(M;\mathbb{F}_2) \cong \mathbb{F}_2$.

orient. non-orient

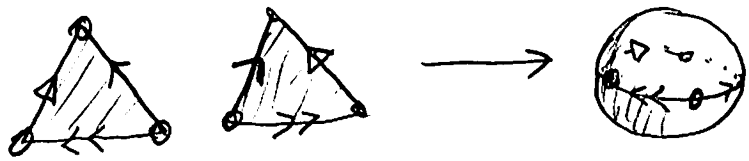
[This is the $k = n$ case of P.D.]

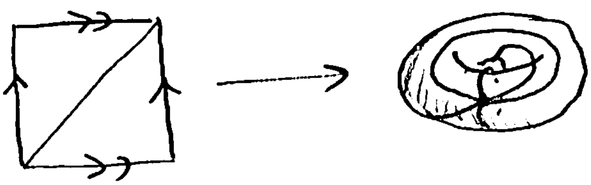
<u>Def</u>: A triangulation of M is a Δ-complex str consisting of n-simplices w/ their $(n-1)$ faces glued together in pairs.

Ex: n = 1

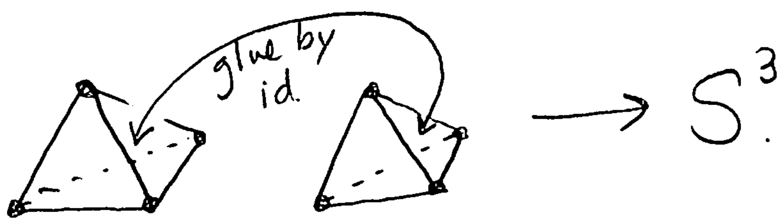


n = 2





n = 3



Claim:

Suppose M has a triangulation. Then $H_n(M; \mathbb{F}_2) = \mathbb{F}_2$.

Q: What is the generator?

Q: Why connected?

Q: Why compact?