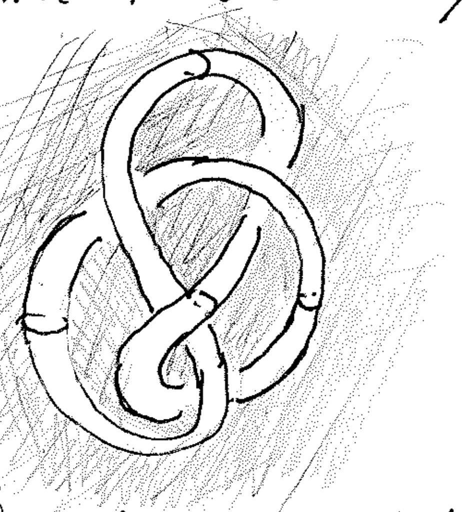
Lecture 1: Foliations and the topology of 3-manifolds

Grad topics class, Fall 2021.

Typical object: Compact orient 3-manifold, either closed, such as S^3, $T^3 = S^1 \times S^1 \times S^1$, $L(p,q)$ (Surface) $\times S^1$, UT(Surface), or with torus boundary, e.g. $S^3 \setminus \mathring{N}(\text{Knot})$



Geometric topology [study of mflds]

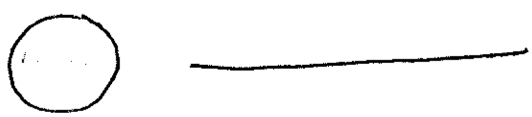
High dim (≥ 5)

Well understood / impossible

↳ turned into homotopy theory

Every finitely-presented G is $\pi_1\left(\begin{matrix}\text{smooth clsd} \\ n\text{-mfld}\end{matrix}\right)$ for all $n \geq 4$.

Low dim (≤ 4)

Dim 1: 

Dim 2:  ... [Classification...]

Dynamics/Moduli: $MCG(\Sigma) = \pi_0(\mathrm{Diff}^+(\Sigma))$

$\mathcal{T}(\Sigma) =$ hyperbolic structures (dim $6g-6$)

Dim 4: π_1 arbitrary, crazy stuff happens: uncountably many nondiffeomorphic smooth mflds that are homeo to $\mathbb{R}^4$! But: Closed M^4 with $\pi_1 = 1$ are det. by cup product on $H^2(M;\mathbb{Z})$. [Geometry rare.]

Dim 3: "Topology is geometry" (Thurston / Perelman)

All closed M^3 have "geometric decompositions" with a "generic" piece having a hyperbolic metric (const curve -1); such are unique (Mostow).

Cor: $\pi_1 M^3$ is residually finite: $\displaystyle\bigcap_{[\pi_1 M : H] < \infty} H = \{1\}$.

[Interested? Stay for 595 AT3!]

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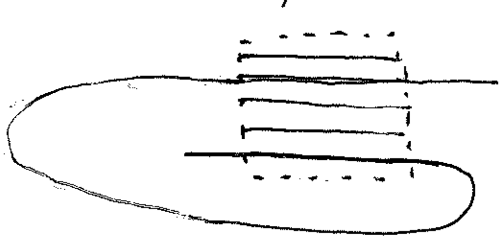
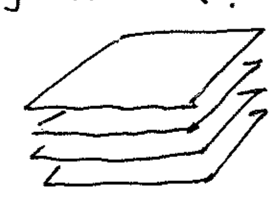
Foliations: M^n smooth
$\mathcal{F}$ partition of M into subsets (leaves) locally like $\mathbb{R}^k \times \{pt\}$ in $\mathbb{R}^n$.

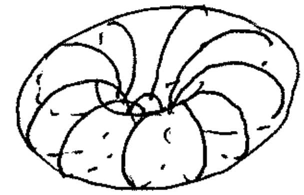




<u>Examples:</u> 1) $M = \mathbb{R}^n$, $\mathcal{F} = \coprod_{y \in \mathbb{R}^{n-k}} \mathbb{R}^k \times \{y\}$

2) $M^n = A^k \times B^{n-k}$, $\mathcal{F} = \{A \times \{b\} \mid b \in B\}$

$A = B = S^1$



3) $M \xrightarrow{f} B$ a smooth submersion

$\mathcal{F} = \{f^{-1}(b) \mid b \in B\}$ [Includes (2)]

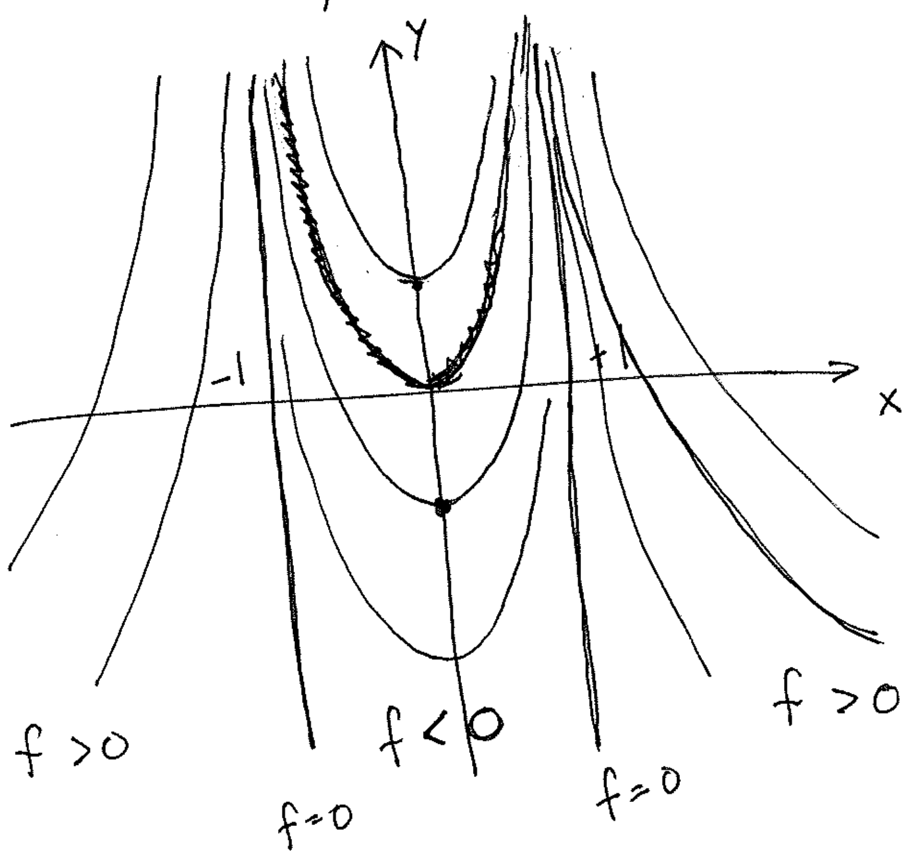
<u>Ex:</u> $f: \mathbb{R}^2 \to \mathbb{R}$ $f(x,y) = (x^2-1)e^y$

$f_x = 2xe^y$ $f_y = (x^2-1)e^y$

<u>$f=0$</u>: $x = \pm 1$

<u>$f=-1$</u>: $e^y = \dfrac{1}{1-x^2}$

<u>$f=1$</u>: $e^y = \dfrac{1}{x^2-1}$



Note

$f(x, y+s) = e^s f(x,y)$

So shifting

$\{f = c\}$ up by s

takes it to $\{f = e^s c\}$

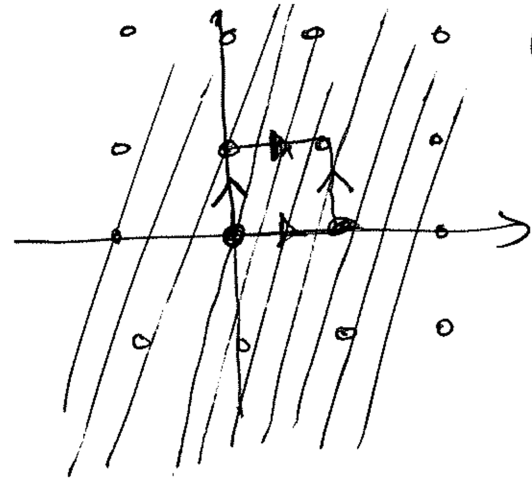
[<u>Note</u>: Not a product or fiber bundle.]

4) $M = \mathbb{R}^2/\mathbb{Z}^2$, $a \in \mathbb{R}$.

$\mathbb{R}^2 \supseteq \tilde{\mathcal{F}} = \{\text{lines of slope } a\}$

preserved by $\mathbb{Z}^2 \Longrightarrow \mathcal{F}$ on M

If $a \notin \mathbb{Q}$, every leaf of $\mathcal{F}$ is dense.



5) A vector field X on M^n that is never 0 has flow lines forming a 1-d foliation. $(\Rightarrow \chi(M) = 0)$.

We'll focus on 2D fol of M^3.

a) Every clsd orient M^3 has such of foliation.

b) $\mathcal{F}$ is <u>taut</u> when $\exists$ a transverse closed loop meeting each leaf.

$\Rightarrow M = S^2 \times S^1$, $\mathbb{RP}^3 \# \mathbb{RP}^3$, or $\tilde{M}_{univ} = \mathbb{R}^3$

(π_1 is infinite)

When co-orientable

$\Rightarrow$ Every cpt leaf L is the simplist surface in its homology class.

$\Rightarrow$ M is a "contact bdry comp" of some symplectic W^4.

$\Rightarrow$ $\widehat{\text{Floer}}_{red}(M) \neq 0$

$\Rightarrow$ $\pi_1 M$ has a total order inv. under left mult.
Conj

<u>Focus:</u> a) Examples / Constructions

b) Broad picture / Big ideas

<u>Not included:</u> Many detailed proofs.

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Go over syllabus, etc....