Leture 2: Definition of a foliation

Idea from last time:

M^n smooth $\mathcal{F}$ = partition of M into k-dim leaves, locally trivial 

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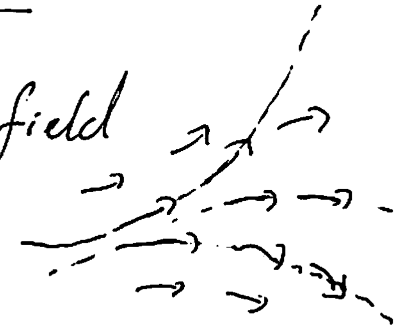


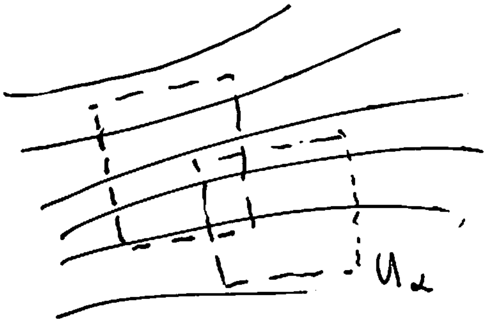
Note: If X is a nowhere vanishing vector field on M^n, then the flow lines of X are a 1-dim foliation. M has such a vector field $\iff \chi(M) = 0 \Leftarrow n$ odd.

Focus: 2-d fol of M^3 [Will do defs only in this setting.]

Fact: Every topological M^3 has a unique smooth structure.

Def: A foliated chart is $U \underset{\text{open}}{\subseteq} M^3$ with a diffeo $\varphi : U \to \mathbb{R}^2 \times \mathbb{R}$. Each $\varphi^{-1}(\mathbb{R}^2 \times \{t\})$ is called a plaque.

Def: A foliation $\mathcal{F}$ is a partition $\{L_\lambda\}_{\lambda \in \mathcal{L}}$ of M^3 into connected immersed surfaces L_λ such that there is an atlas $(U_\alpha, \varphi_\alpha)_{\alpha \in \mathcal{A}}$ of foliated charts where every $L_\lambda \cap U_\alpha$ is a union of plaques of U_α.



Note: If V is a connected component of $U_\alpha \cap U_\beta$, then

$$\varphi_\beta \circ \varphi_\alpha^{-1} : \varphi_\alpha(V) \longrightarrow \varphi_\beta(V)$$

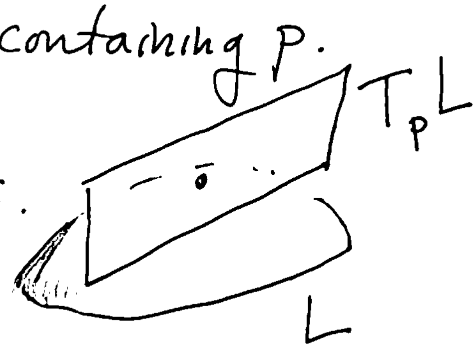
has the form $(x, y, t) \longmapsto (a(x,y,t),\ b(x,y,z),\ c(t))$

transverse coor.

Can define a foliation from an atlas of charts whose transition fns have this form. [Fol I, Chap 1.2]

A *plane field* ξ on M^3 is a choice of 2-plane ξ_p in T_pM for all $p \in M$. It's *smooth* when locally $\xi = \ker(\alpha)$ for a smooth 1-form α.

Ex: For a fol $\mathcal{F}$ of M^3, take $\xi = T\mathcal{F}$, i.e. $\xi_p = T_pL$ where L is the leaf containing p.



Note: In a fol chart, take $\alpha = dt$.

Def: ξ is *integrable* when $\exists\ \mathcal{F}$ with $T\mathcal{F} = \xi$

<u>Frobenius Thm</u>: ξ is integrable $\Longleftrightarrow$ for all α where $\xi = \ker(\alpha)$ locally, then $\alpha \wedge d\alpha = 0$.

<u>Note</u>: In a fol chart, $\alpha(\partial/\partial x) = \alpha(\partial/\partial y) = 0$ at any point, so $\alpha = f(x,y,t)\,dt \Rightarrow \alpha \wedge d\alpha = 0$.

[More standard form: the set of vector fields tang. to ξ is closed under the Lie bracket.]

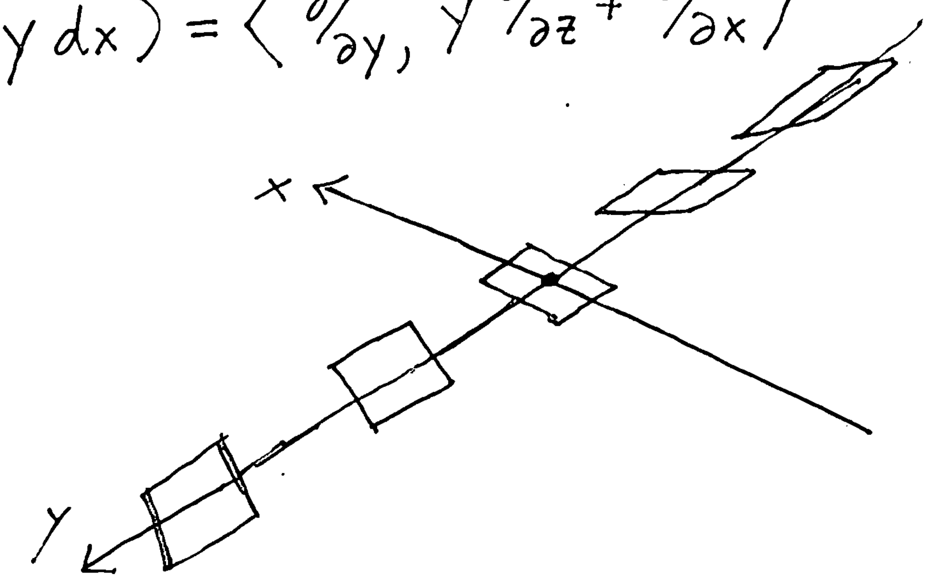
<u>Pf</u>: [§4.1.3 of Cal] or [§1.3 of Fol I].

<u>Def</u>: ξ is a <u>contact structure</u> when every local α has $\alpha \wedge d\alpha$ nowhere 0.

<u>Ex</u>: $\xi = \ker(dz - y\,dx) = \langle \partial/\partial y,\ y\,\partial/\partial z + \partial/\partial x \rangle$

<u>Idea of why this isn't integrable</u>:

1) Zoom so leaf L through 0 is nearly flat.



2) Take a closed c on L that looks like a

circle viewed from above.

3) c is tangent to ξ at each point, in particular c' has z-comp ≥ 0 and > 0 when $y \neq 0$. Contradicts that c is closed.

<u>Note</u>: ξ is a connection on the bundle $\begin{array}{ccc} \mathbb{R}^3 & & (x,y,z) \\ \downarrow & & \downarrow \\ \mathbb{R}^2 & & (x,y) \end{array}$ but its not flat having nontrivial holonomy.

[Eliasberg - Thurston] If $\mathcal{F}$ is a foliation of a clsd M^3 then $T\mathcal{F}$ can be C^0-perturbed to a contact str ξ.

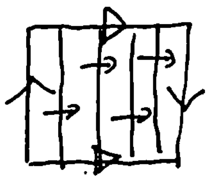
<u>Note</u>: If a contact ξ is $\ker(\alpha)$ locally then either $\underbrace{\alpha \wedge d\alpha > 0}$ or $\alpha \wedge d\alpha < 0$ everywhere.

$$\alpha \wedge d\alpha = f(x,y,z)\, dx \wedge dy \wedge dz$$

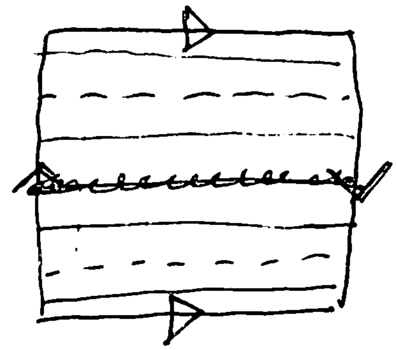
Def: $\mathcal{F}$ is co-orientable when $\exists$ a vector field X on M with $X_p \notin T_p\mathcal{F}$ for all p. ($\Rightarrow X_p \neq 0$).

$\Longleftrightarrow$ $\exists$ a global 1-form with $\ker(\alpha) = T\mathcal{F}$.

Ex: M = Klein bottle



Non Ex:



Notes: If M^3 is orient, then $\mathcal{F}$ co-orient $\Longleftrightarrow$ all leaves are consist. orient.

If $\mathcal{F}$ is not co-orient, then M has a 2-fold cover M' s.t. $\mathcal{F}'$ is co-orient.

Next time: Constructions / Examples, including fol of S^3.