Lecture 3: Some examples

Last time: ξ plane field on M^3

$\xi = T\mathcal{F} \iff \alpha \wedge d\alpha = 0$ if $\ker(\alpha) = \xi$ locally

$\xi =$ contact structure $\iff \alpha \wedge d\alpha$ nowhere 0.

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Some contact strs: 1) $\mathbb{R}^3$, $\xi = \ker(dz - y\,dx)$

2) $S^3 \subseteq \mathbb{C}^2$, $\xi_p =$ unique complex line in $T_p S^3$

$(x_1 + iy_1, x_2 + iy_2)$ $\qquad \alpha = \sum_{j=1}^{2} x_j\,dy_j - y_j\,dx_j$

3) $T^3 = \mathbb{R}^3 / \mathbb{Z}^3 \qquad \alpha = \cos(2\pi z)\,dx - \sin(2\pi z)\,dy$

[For (M^n, g), T^*M is symplectic, UT^*M contact]

[Eliasberg–Thurston] If $\mathcal{F}$ is a foliation of a clsd M^3, then $T\mathcal{F}$ can be C^0-perturbed to a contact str.

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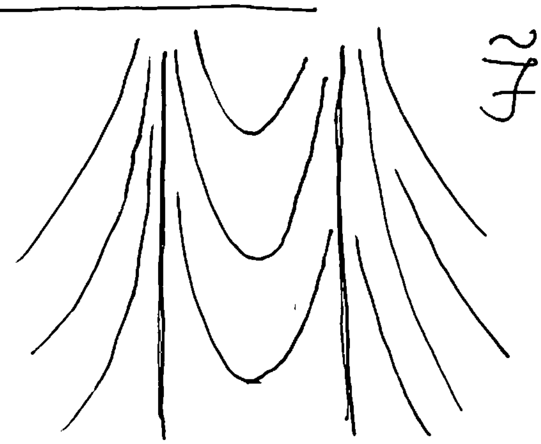
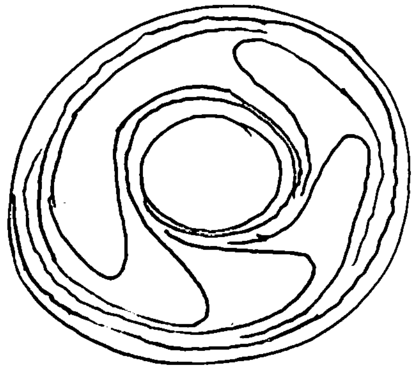
Recall: $f(x,y) = (x^2 - 1)e^y$ a submersion. $\tilde{\mathcal{F}}$ is inv. under vertical trans.



Set $A = [-1,1] \times \mathbb{R} \big/ (x,y) \longmapsto (x, y+1)$

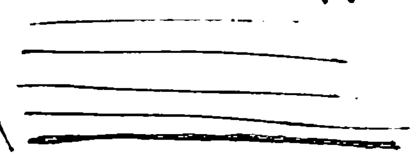
This is the <u>Reeb foliation</u> of an annulus.



<u>Foliations</u> of <u>mflds</u> w/ <u>∂</u>:

tangential: ∂M is a leaf

<u>Ex</u>: $L = \mathbb{R} \times \{y\}$

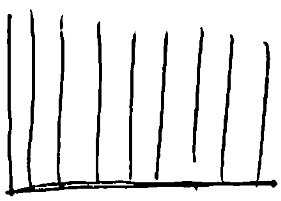


∂M

$M = \{(x,y) \mid y \geq 0$

transverse: leaves perp. to ∂M.

<u>Ex</u>:

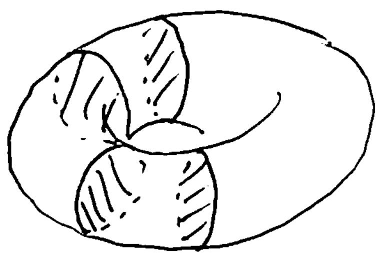


$L = \{x\} \times [0, \infty)$

<u>Ex</u>: $M = D^2 \times S^1$

solid torus

$\mathcal{F} = \{ D^2 \times \{pt\} \}$

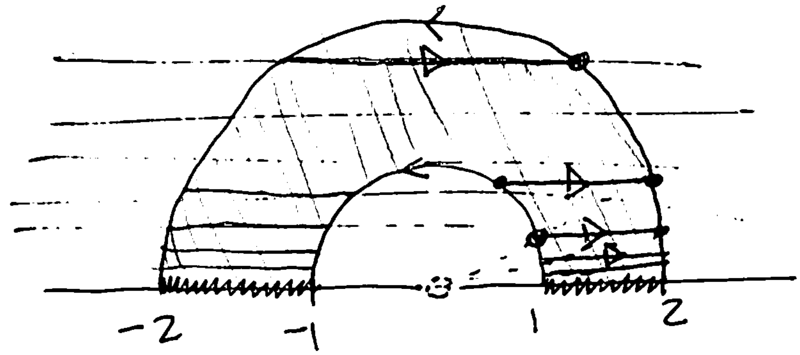


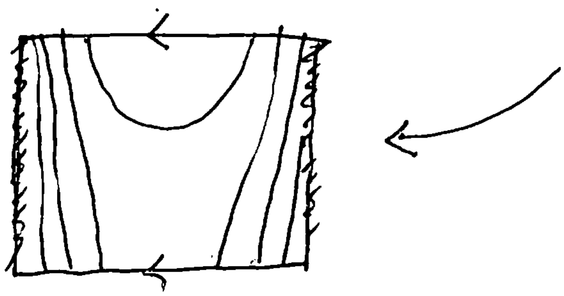
Another const of A: Set $\tilde{A} = \mathbb{R} \times [0, \infty) - \{(0,0)\}$

and $A = \tilde{A} \big/ (x,y) \mapsto (2x, 2y)$

$\tilde{\mathcal{F}}$ is invariant, decends to $\mathcal{F}$ on A

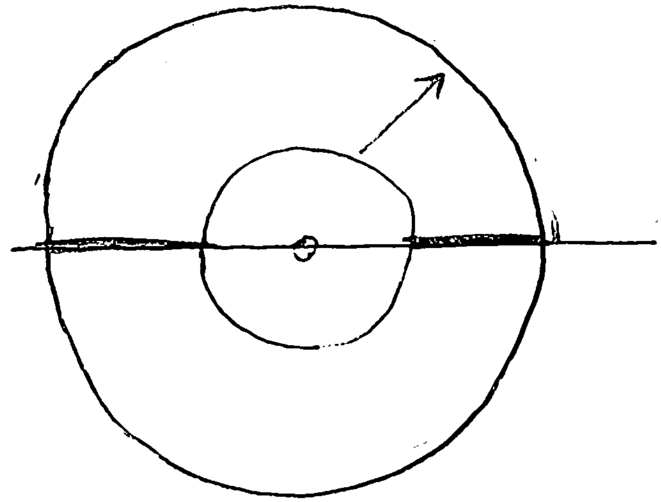
$\tilde{\mathcal{F}} = \mathbb{R} \times \{y\}$

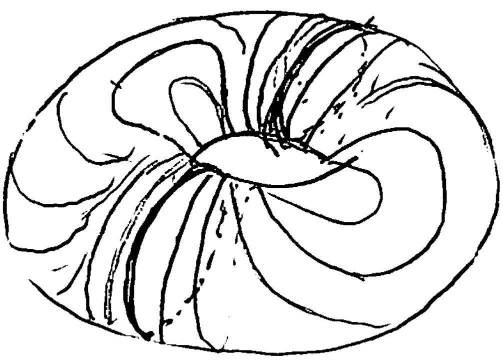




Ex: $\tilde{T} = \mathbb{R}^2 \setminus \{0\}$

$T = \tilde{T} / (x,y) \to (2x, 2y)$



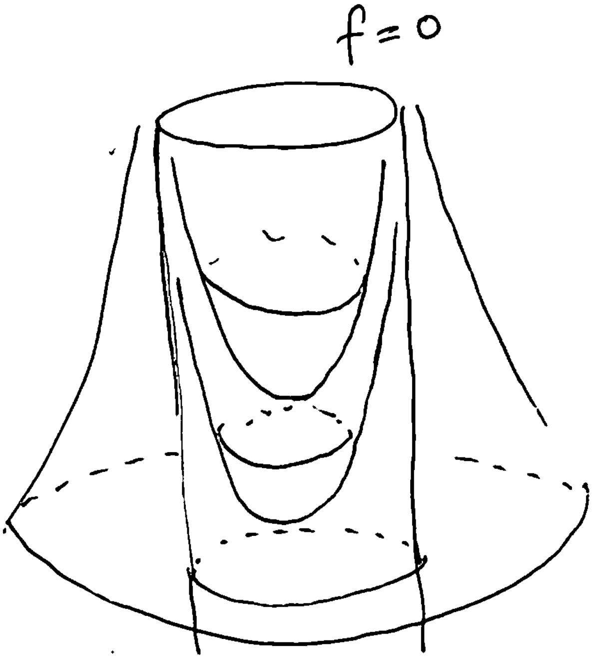


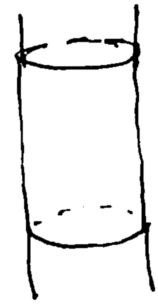
Has exactly two compact leaves, every other leaf spirals out towards them. [Contrast with fol from Lect 1]

Ex: Reeb solid torus.

x^2+y^2

Consider $f(x,y,z) = (r^2-1)e^z$, a submersion $\mathbb{R}^3 \to \mathbb{R}$. Get res. of rotating earier example about vertical axis:





$\tilde{M} = \{ r \leq 1 \}$

$M = \tilde{M} / _{(x,y,z) \mapsto (x,y,z+1)}$

$\cong D^2 \times S^1$

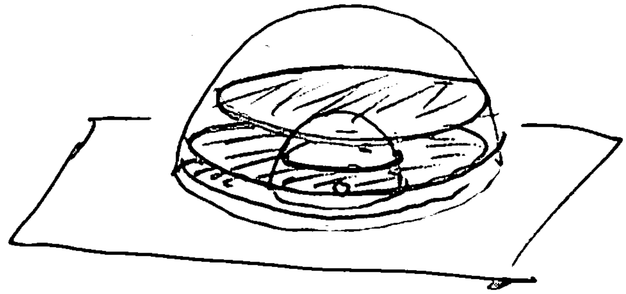
<u>Aside</u>: $\mathcal{F}$ does <u>not</u> come from a submersion $M \to \mathbb{R}$.

Get $\mathcal{F}$:

$\mathcal{F} = \{ \mathbb{R}^2 \times \{z\} \}$

Same as: $\tilde{N} = \mathbb{R}^2 \times [0, \infty) \setminus \{0\}$

$N = \tilde{N} / _{\vec{v} \to 2\vec{v}}$



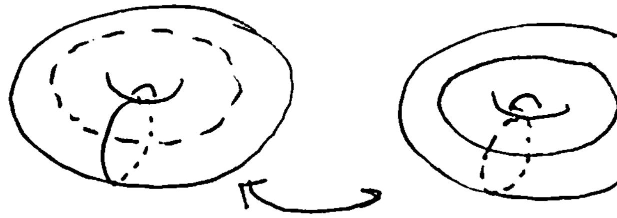
<u>Ex</u>: $\tilde{M} = \mathbb{R}^3 \setminus \{0\}$

$M = \tilde{M} / _{\vec{v} \to 2\vec{v}} = S^2 \times S^1$

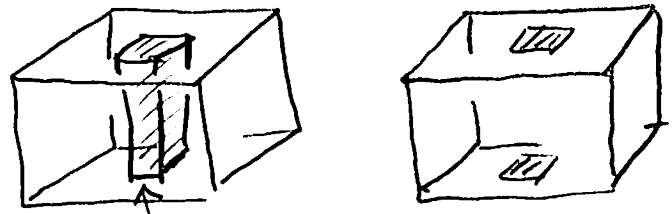
$\mathcal{F}$ is two Reeb components "doubled" across the ∂.

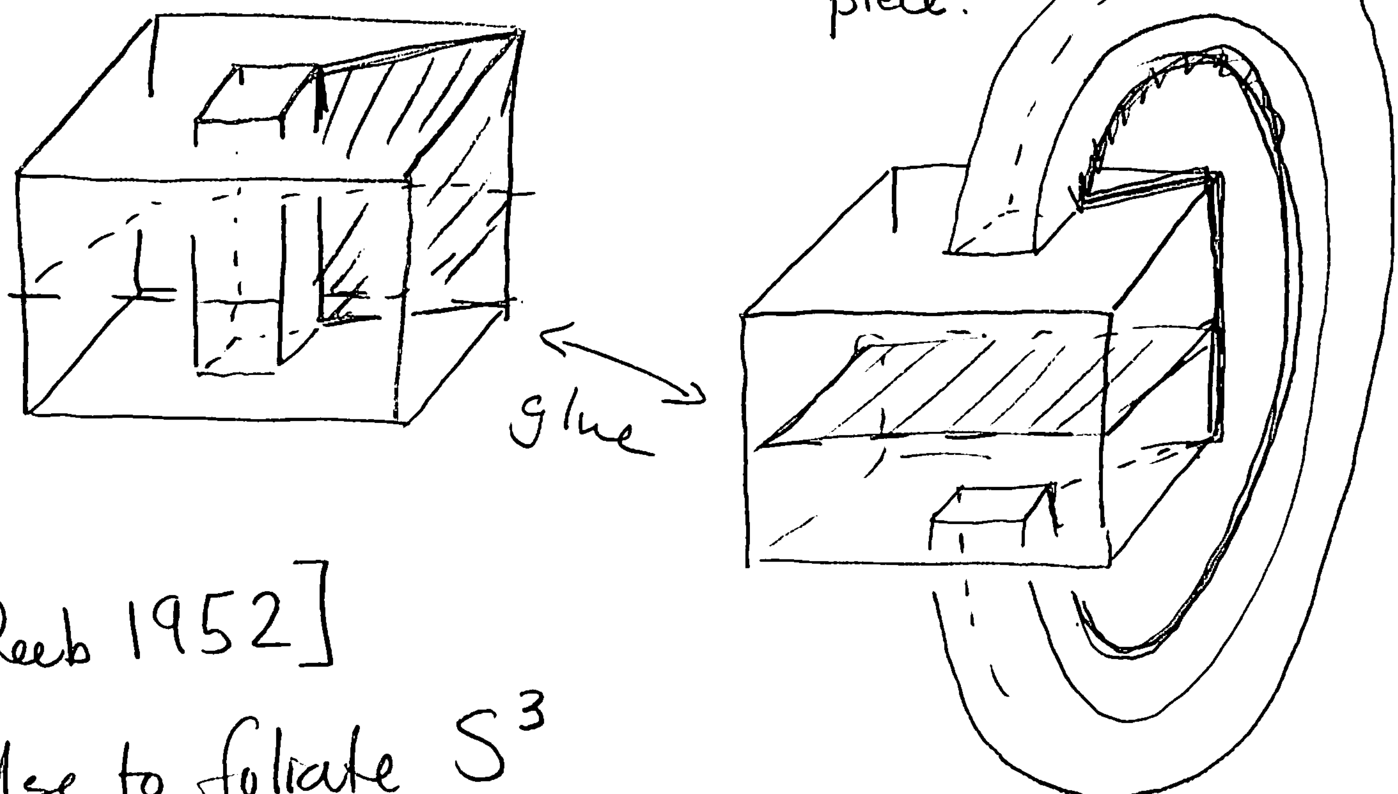
[Very diff from the product fol.]

S^3 = union of 2 solid tori



Pf: S^3 = two balls





[Reeb 1952]

Use to foliate S^3 via two Reeb solid tori as shown on the next page.

Q: Is this really smooth along the cpt leaf (all other leaves are planes)?
How do we think about this?

A. Holonomy!