Lecture 17: More on the L-space Conjecture.

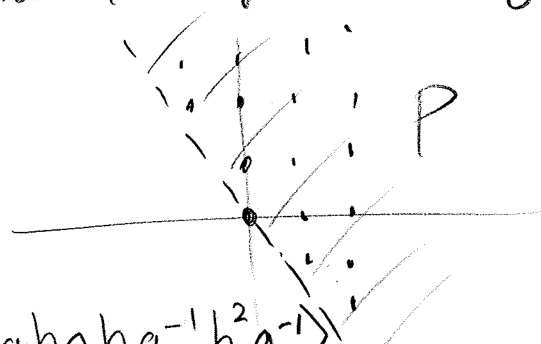
Conj: M^3 clsed orient irred. TFAE

1) M has a co-orient taut fol.
2) $\pi_1 M \hookrightarrow \mathrm{Homeo}^+(\mathbb{R}) \iff \pi_1 M$ is left-orderable.
3) M is not an L-space, i.e. $\widehat{HF}_{red}(M) \neq 0$.

Prop: G is left-orderable $\iff G = P \amalg \{1\} \amalg P^{-1}$
with $P \cdot P \subseteq P$.

Pf ($\Rightarrow$) Set $P = \{g \in G \mid g > 1\}$. Note
$g > 1 \iff 1 > g^{-1}$ and $g, h > 1$ then $g \cdot h > g \cdot 1 > 1$.
($\Leftarrow$) Say $g > h \iff h^{-1} g \in P$. $\blacksquare$

Ex: Each line with irrational slope in $\mathbb{R}^2$ gives a left-order on $\mathbb{Z}^2$



Ex: $G = \pi_1(\text{Weeks mfld})$

$$= \langle a, b \mid \underbrace{ababab^{-1}a^2b^{-1}}_{R_1}, \underbrace{bababa^{-1}b^2a^{-1}}_{R_2} \rangle$$

[Smallest volume hyp 3-mfld $0.94\ldots$]

is not left orderable.

Pf: Can assume $a \in P$. If $b^{-1} \in P$, then

$b^{-1} R_2^{-1} b \cdot R_1 = b^{-1} a b^{-1} a^{-2} b^{-2} a^2 b^{-1} a^2 b^{-1} \in P$, a contra. since $1 \notin P$.

Assume $b \in P$. [Hiding: $a, b, ab^{-1} \neq 1$]

- If $ab^{-1} \in P$, then so is $abab(ab^{-1})a(ab^{-1})$.
- If $ba^{-1} \in P$, then so is $baba(ba^{-1})b(ba^{-1})$.

In both cases, $1 \in P$ a contradiction. ▨

[A non-left orderable G which is finitely gen with solvable word problem has a proof of non-orderability "like this".]

Thm: Suppose M^3 is clsd orient irred. If $\pi_1 M$ has a non-trivial homom to $\mathrm{Homeo}^+(\mathbb{R})$ then $\pi_1 M$ is left-orderable.

Pf: Cor of: every f.g. subgp of a f.g. G has a hom. to $\mathrm{Homeo}^+ G \Rightarrow G$ is left-orderable. ▨

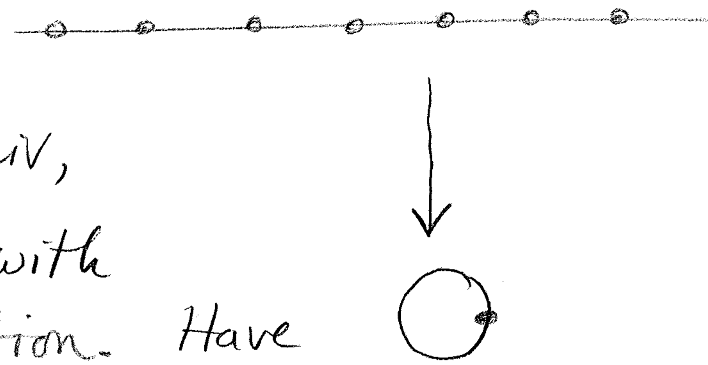
Cor: The Weeks mfld does not have a taut fol with $\widetilde{\mathcal{L}} \cong \mathbb{R}$.

Pf: Since $H_1 \cong (\mathbb{Z}/5)^2$, any fol is co-orient. So get $\pi_1 M \longrightarrow \mathrm{Homeo}^+(\mathbb{R})$ from action on leaf space. So $\pi_1 M$ is left-orderable by thm, a contradiction.

Basic tool:

$$\rho : \pi_1 M \longrightarrow PSL_2\mathbb{R} \leq \mathrm{Homeo}^+(S^1)$$

Define $\widetilde{\mathrm{Homeo}^+(S^1)} \leq \mathrm{Homeo}^+(\mathbb{R})$ to be the subgroup of lifts of elts of $\mathrm{Homeo}^+(S^1)$. Equiv, is those elts commuting with action of $\mathbb{Z}$ by translation. Have



$$0 \longrightarrow \mathbb{Z} \longrightarrow \widetilde{\mathrm{Homeo}^+(S^1)} \longrightarrow \mathrm{Homeo}^+(S^1) \longrightarrow 1$$

a central extension. Given $\rho : \pi_1 M \to \mathrm{Homeo}^+(S^1)$, can lift to $\widetilde{\mathrm{Homeo}^+(S^1)}$ iff $e(\rho) \in H^2(M;\mathbb{Z}) = 0$. where $e(\rho)$ is the Euler class (see Fol II, Ch 4).

Remark: ρ lifts to $\widetilde{\mathrm{Homeo}^+(S^1)} \iff$ the (flat) circle bundle over M assoc. to ρ has a section. For the unit circle bundle of a $\mathbb{C}$-line bundle, $e = c_1$.

Cor: If $H_1(M;\mathbb{Z}) = 0$ and M has a taut fol, then $\pi_1 M$ is left-orderable. Pf: Univ. Circle.

<u>Thm</u>: The Weeks mfld does not have a taut fol.

<u>Pf</u>: Since $H_1(W;\mathbb{Z}) = (\mathbb{Z}/5)^2$, the Euler class of $\rho_{univ} : \pi_1 W \hookrightarrow \mathrm{Homeo}^+(S^1_\infty)$ has order 1 or 5 in $H^2(W;\mathbb{Z})$. Can't have $e(\rho_{univ}) = 0$ since $\pi_1 W$ is not left-order. Hence there exists an index 5 subgp $\Gamma \leq \pi_1 W$ so $e(\rho_{univ}|_\Gamma) = 0$ and so Γ is left order. A similar but more involved arg. shows this is imposs. See [CD 2003] ▨

Let $\widetilde{PSL_2\mathbb{R}}$ be the preimage of $PSL_2\mathbb{R}$ in $\widetilde{\mathrm{Homeo}^+}(S^1)$. It is also the univ. covering Lie gp of $PSL_2\mathbb{R} \cong UT(\mathbb{H}^2) \simeq_{h.e.} S^1$. So

$$0 \longrightarrow \mathbb{Z} \longrightarrow \widetilde{PSL_2\mathbb{R}} \longrightarrow PSL_2\mathbb{R} \longrightarrow 1$$

Any hyp M^3 has $\pi_1 M \hookrightarrow \mathrm{Isom}^+(\mathbb{H}^3) \cong PSL_2\mathbb{C}$

By local rigidity, image conjugate into $PSL_2 K$ for a number field K. If K has a real embedding, get $\pi_1 M \leq PSL_2\mathbb{R}$.

For the 300k mfld sample, found an average of 7.9 $SL_2\mathbb{R}$ reps per mfld.
Enough have $e = 0$ that some 64,000 have left-orderable π_1. All were non-L-spaces, const. w/ conjecture. If $e(\rho) = 0$ with prob $1/|H^2(M;\mathbb{Z})|$ except 6,300 counterexamples ($p \approx 10^{-2,700}$)

[Boyer-Hu] $e(\rho_{univ}) = e(T\mathcal{F})$ for a co-orient taut fol.

Can use to show > 32,000 of these mflds have orderable π_1; more than 160,000 have taut fol.