Lecture 9: 3-manifolds via Dehn surgery.

Goal: Every clsd orient $N^3 = M_f(\alpha_1, \alpha_2, .., \alpha_n)$ for some diff f of a surface F.

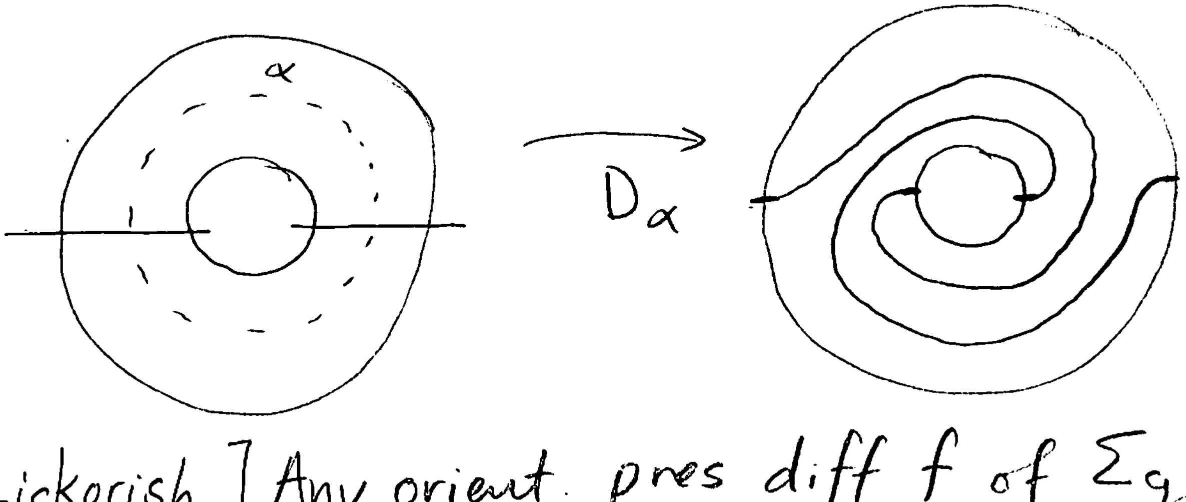
So far: Every such N has a Heegaard splitting, i.e.

$N = H_g \cup_f H_g$ where H_g = handlebody [g dep. on M.]

f a diff of $\partial H_g = \Sigma_g$.

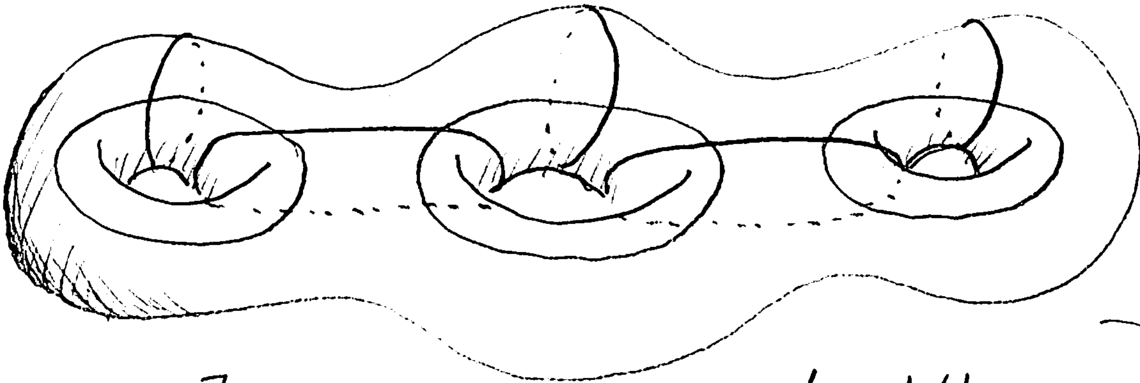
[Need to understand the poss f in $H_g \cup_f H_g$.]

Suppose α is an ess simple closed curve on Σ. The pos. Dehn twist D_α is the diffeo of Σ that does this on an annulus nbhd of α and = id elsewhere.



[Lickorish] Any orient. pres diff f of Σ_g is isotopic to a composition of Dehn twists along the following $3g-1$ curves.

[You many have to use each curve many times.]



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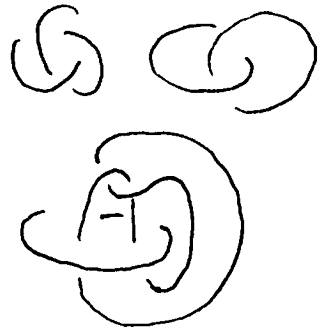
[Maher 2010] A random Heegaard splitting is hyperbolic with prob $\longrightarrow 1$.

[D-Thurston 2006] Fix n. Then $\mathbb{E}(\#\text{ of reg. } A_n\text{-covers}) \to 1$ as $g \to \infty$.

A <u>link</u> is an embedding $\coprod S^1 \hookrightarrow S^3$



Its <u>exterior</u> is $S^3 \setminus \mathring{N}(L)$ which has $\# L$ tori as its boundary.

A Dehn filling of the exterior is called a <u>Dehn surgery</u> on L.

<u>Thm</u>: Every clsd orient M^3 is Dehn surgery on a link L in S^3.

<u>Cor</u>: Every such $M^3 = \partial W$ for some smooth simply connected W^4.

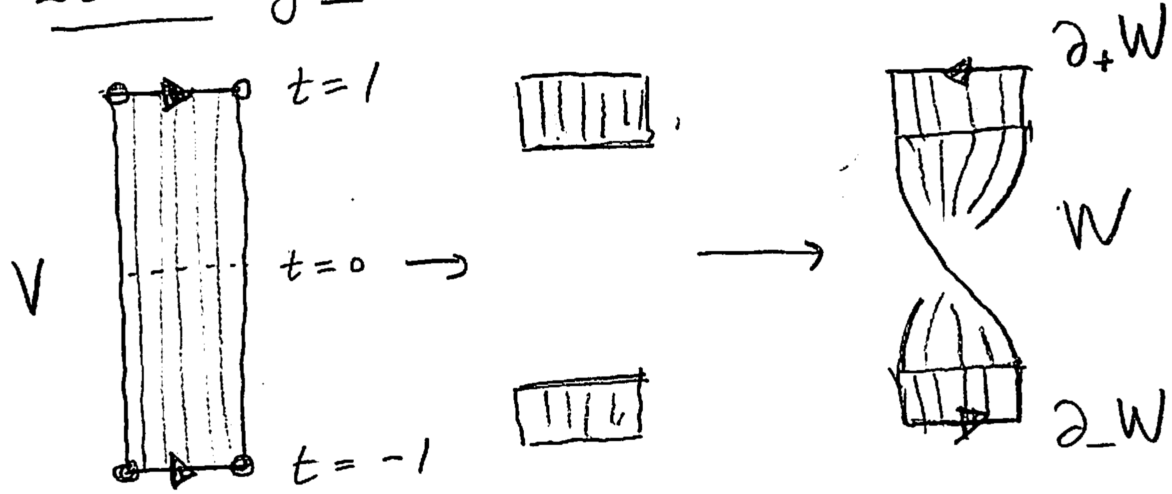
[*Note*: Cor holds for surfaces too!]

Key idea: α a simple closed curve on Σ.

$$V = \Sigma \times [-1,1] \qquad \tilde{\alpha} = \alpha \times \{0\}.$$

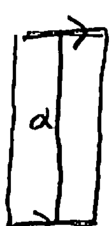
Lemma: There is a Dehn surgery on $\tilde{\alpha}$ resulting in W with $W \cong \Sigma \times [-1,1]$ so that the induced map $\Sigma = \partial_- W \longrightarrow \partial_+ W = \Sigma$ is D_α.

2D analogue:

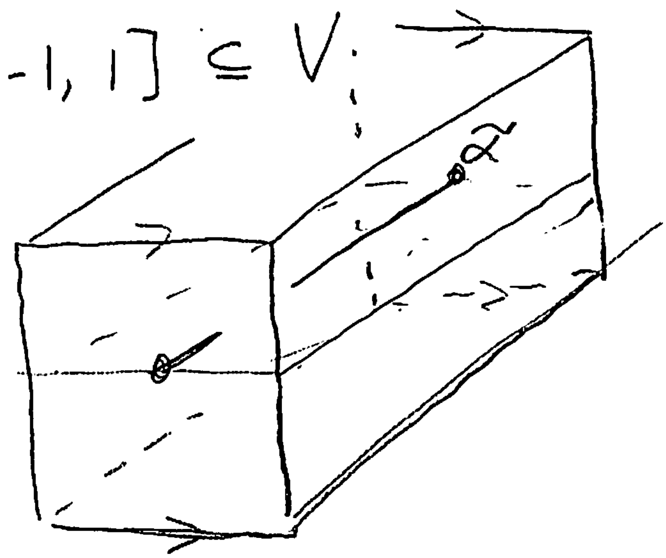


$\Sigma =$ 

Pf of Lemma: Let A be a reg nbhd of α in Σ

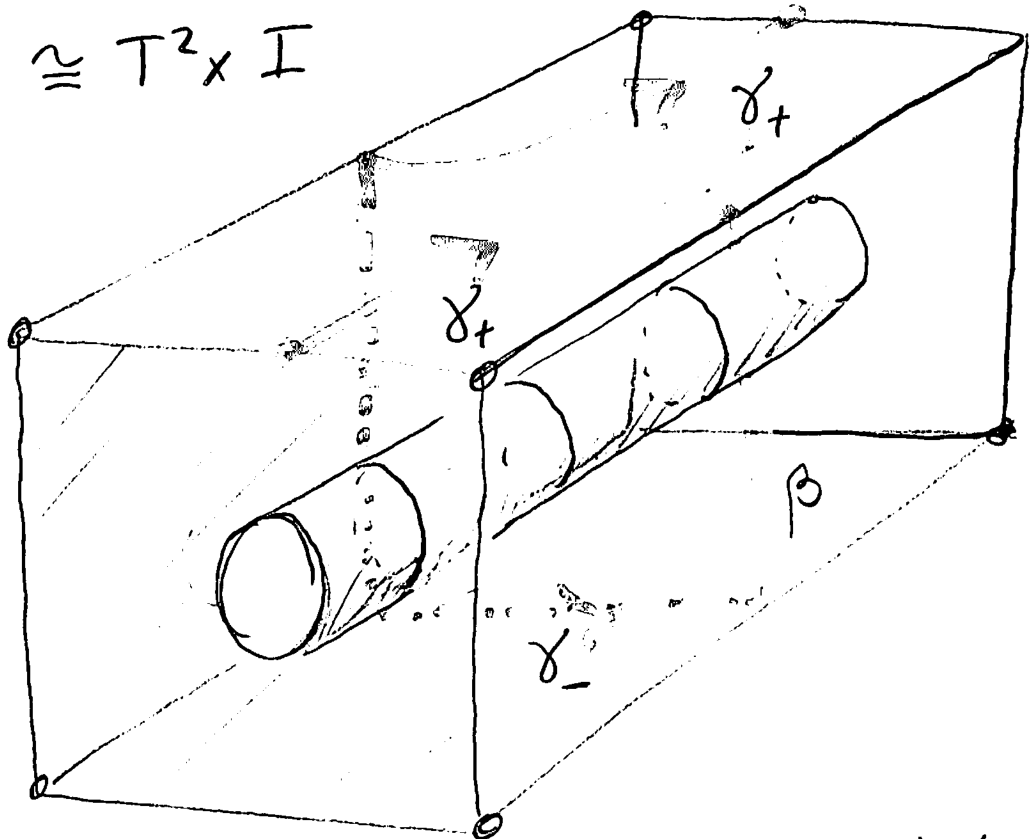


Consider $A \times [-1,1] \subseteq V$:



Remove $\mathring{N}(\tilde{\alpha})$ from $A \times [-1,1]$ to get B:

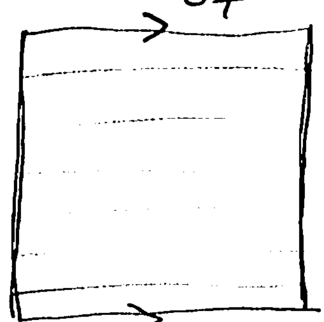
This is $\cong T^2 \times I$



Thus Dehn filling B results in a solid torus C.

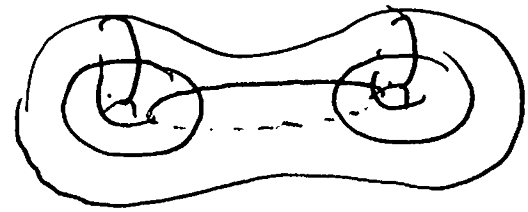
Do the Dehn filling so that $\beta = \partial\left(\begin{array}{c}\text{meridian}\\ \text{disk in}\\ D^2 \times S^1\end{array}\right)$.

Pushing across meridian disk shows γ_- becomes γ_+ in the product str on C so C implements D_α. ▨

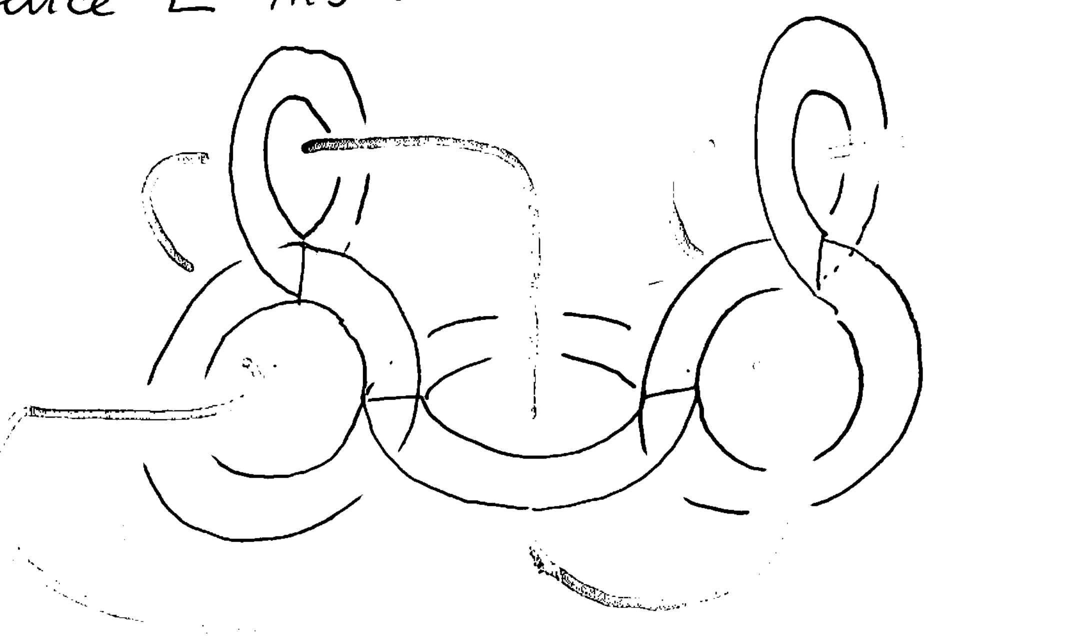


Pf of Thm: Let H_g be the std handlebody in S^3. Let f_0 be such that $H_g \cup_{f_0} H_g$ is the assoc Heegaard splitting. To create $M = H_g \cup_f H_g$, suppose $f_0^{-1} \circ f = D_{\alpha_n}^{\pm 1} \circ \cdots \circ D_{\alpha_1}^{\pm 1}$ [where α_i are Lickorish std curves]. Let $N \cong \Sigma_g \times (0, n+1)$ be a nbhd of $\Sigma_g = \partial H_g$ in S^3. Set $L = \dot{\bigcup} \alpha_i \times \{i\}$. Then the appropriate Dehn surgery on L gives the manifold corresp. to $f_0 \circ D_{\alpha_n}^{\pm 1} \circ \cdots \circ D_{\alpha_2}^{\pm 1} \circ D_{\alpha_1}^{\pm 1} = f$. ▨

Proof of Goal: Suppose $g = 2$ so $\alpha_1, \ldots, \alpha_5$ are



Hence L lies on



Add in this extra component to get $\tilde{L}$.

Exterior of new component is $D^2 \times S^1$.

Moreover, each component of L is trans. to the prod. fol on $D^2 \times S^1$. Thus

$S^3 \setminus \mathring{N}(\tilde{L})$ is fibered $\left(F = D^2 \setminus \mathring{N}(\# L \text{ pts}) \right)$

and so our manifold is Dehn filling on some M_f. $\blacksquare$

Note: In fact, this is an open book decomp.

Rec. ref: Rolfsen's Knots and Links (Ch 9 + 10).

Figure 8.1.6 in Fol II is incorrect.