Lecture 10: Surfaces in 3-manifolds

Last time:

Thm: Every closed, connected orient N^3 is Dehn surgery on some link in S^3.

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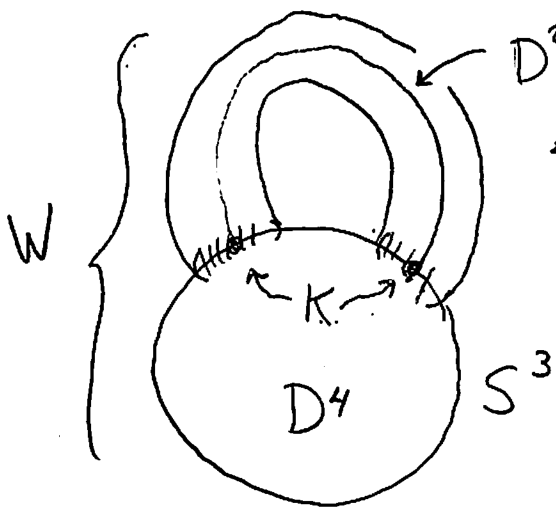
Cor: Every closed orient M^3 is ∂W for some smooth simply connected W^4.

Idea: Reduce to the case that M is connected by taking connect sums of the W's. Suppose N is ± 1 Dehn surgery on a knot K in S^3. [All surgeries last time were of this type.]



$D^2\times\{pt\}$

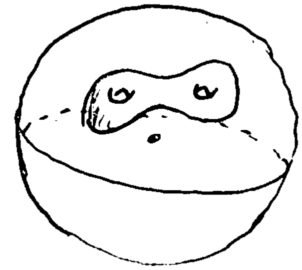
4D 2-handle, i.e. $D^2\times D^2$ glued along $\overbrace{\partial D^2\times D^2}^{\text{solid torus}}$, attached to the 0-handle D^4.

$$\partial W = \left(S^3 \setminus \operatorname{int}(\partial D^2\times D^2)\right) \cup \underbrace{(D^2\times \partial D^2)}_{\text{also a solid torus.}}$$

In particular, ∂W is a Dehn surgery on K in S^3. Same think if M is Dehn surgery on a link L. Note $W \simeq_{h.e.} \vee S^2$.

So $\pi_1 = 1$ ∎

<u>Surfaces in 3-mflds:</u> There are lots of uninteresting ones:



Suppose M^3 clsd orient.

A clsd connect. $F \hookrightarrow M$ is <u>incompressible</u> when

$F \neq S^2$ and $\pi_1 F \longrightarrow \pi_1 M$ is injective.

<u>Ex:</u> $F \times \{pt\}$ in $M = F \times S^1$ as $\pi_1 M = \pi_1 F \times \mathbb{Z}$, $\Big\}\ F \neq S^2$

<u>Ex:</u> F in M_f for any $f: F \circlearrowleft \cong$

<u>Non Ex:</u> Any Heegaard surface, any surface $\subseteq \mathbb{R}^3 \subseteq M$.

<u>Non Ex:</u> S^3 contains no incompressible surfaces.

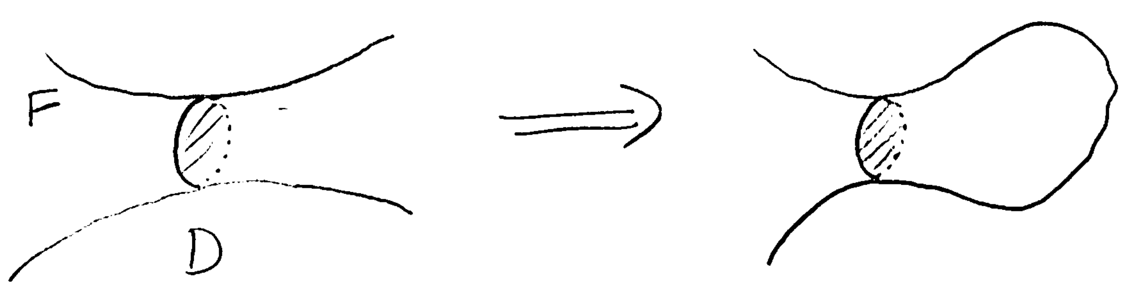
A <u>compressing disc</u> for $F^2 \subset M^3$ is an emb. disc

$D \subset M$ with $D \cap F = \partial D$

A clsd F^2 in M^3 is

<u>geometrically incompress.</u> when $\forall$ comp. discs D

one has ∂D bounds a disc in F.



Ex: Any incomp F is geom. incomp. since

$\partial D = 1$ in $\pi_1 M \Rightarrow \partial D = 1$ in $\pi_1 F \Rightarrow \partial D$ bounds a disc in F.

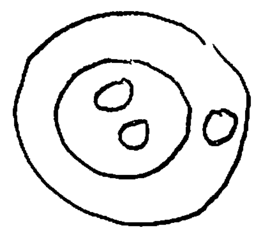
[Papakyriakopoulos 1956] If F is 2-sided ($\Leftrightarrow$ orient as M^3 is orient) and geom. incomp then it is incomp. [Loop thm.]

Cor: If $\pi_1 F \to \pi_1 M$ has kernel, there is an emb. $\alpha \neq 1$ in it.

2 sided

Def: A disconn. $F^2 \subseteq M^3$ is incomp. if every conn. comp. is incomp.

Exercise: $F^2 \subseteq M^3$ is geometrically incomp $\Leftrightarrow$ each conn. comp. is geometricall incomp.



Def: M^3 is prime when $M = A \# B$ $\Rightarrow A \cong S^3$ or $B \cong S^3$.

Thm: Any clsd orient $M^3 = A_1 \# \cdots \# A_K$ where each A_K is prime. [Unique up to order]

Def: M^3 is irreducible when every emb. S^2 is the bdry of a 3-ball.

Ex: $\mathbb{R}^3$, S^3, any M^3 with $\tilde{M}_{univ} = \mathbb{R}^3$ or S^3.

<u>Prop</u>: An irred M^3 is prime. $S^2 \times S^1$ is the only clsd orient 3-mfld which is prime but not irred.

<u>Pf Hint</u>: prime $\Leftrightarrow$ every separating S^2 bounds a ball.

<u>Sphere Thm</u> If $\pi_2 M^3 \neq 0$ then $\exists$ an emb. $S^2 \neq 0$ in π_2.

<u>Cor</u>: M^3 irred $\Rightarrow \pi_2 M = 0$.

<u>Cor</u>: M^3 irred, $\pi_1 M$ infinite $\Rightarrow$ M is a $K(\pi_1 M, 1)$.

A compact orient M^3 is <u>Haken</u> when it is irreducible and contains an orient. incompress. surface.

[Waldhausen 1960s] If N, M are clsd Haken mflds with $\pi_1 N \cong \pi_1 M$ ($\Leftrightarrow N \simeq_{h.e.} M$) then N and M are diffeo.

[True for surfaces, false for lens spaces]

<u>Cor of Geometrization</u>: Holds for any clsd N^3, M^3 that are irred and $|\pi_1| = \infty$.

<u>Borel Conjecture</u>: If N, M are clsd asyperical n-mflds with $\pi_1 N \cong \pi_1 M$, then M and N are homeomorphic.

<u>Virtually Haken Thm</u> [Agol, Wise, ... 2012]
If M^3 is clsd, irred, and $|\pi_1| = \infty$, then M has a finite cover which is Haken.

[If time remains, blather about the above...]