Lecture 6: Proof of Reeb stability.

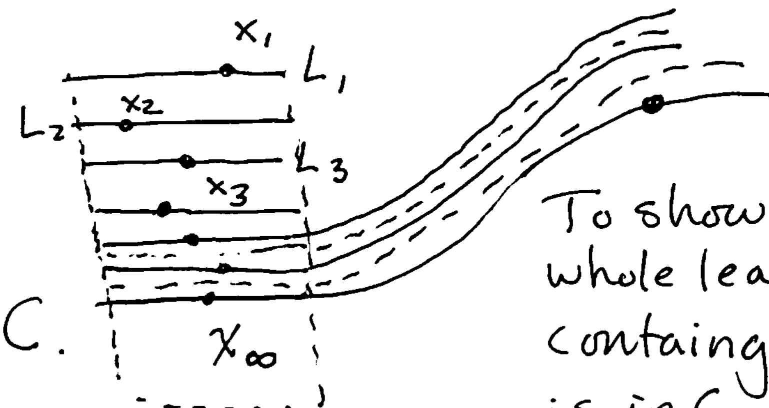
Last time:

Reeb Stability: Suppose a compact leaf L of $\mathcal{F}$ on M^3 has trivial holonomy. Then L has an open nbhd $N \cong L \times (-1,1)$ where $\mathcal{F}$ is the product fol.

Reeb Stability: Suppose $\mathcal{F}$ is a co-orient fol of a clsd conn. orient M^3. If some leaf of $\mathcal{F}$ is S^2, then $M = S^2 \times S^1$ and $\mathcal{F}$ is the product fol.

Lemma 1: Suppose $\{L_\lambda\}_{\lambda \in \mathcal{C}}$ is a collection of leaves of some $\mathcal{F}$. Then the closure C of $\bigcup_{\lambda \in \mathcal{C}} L_\lambda$ is a union of leaves.

Proof: $x_\infty \in M$ is a limit of $x_n \in L_n$. In a product chart:

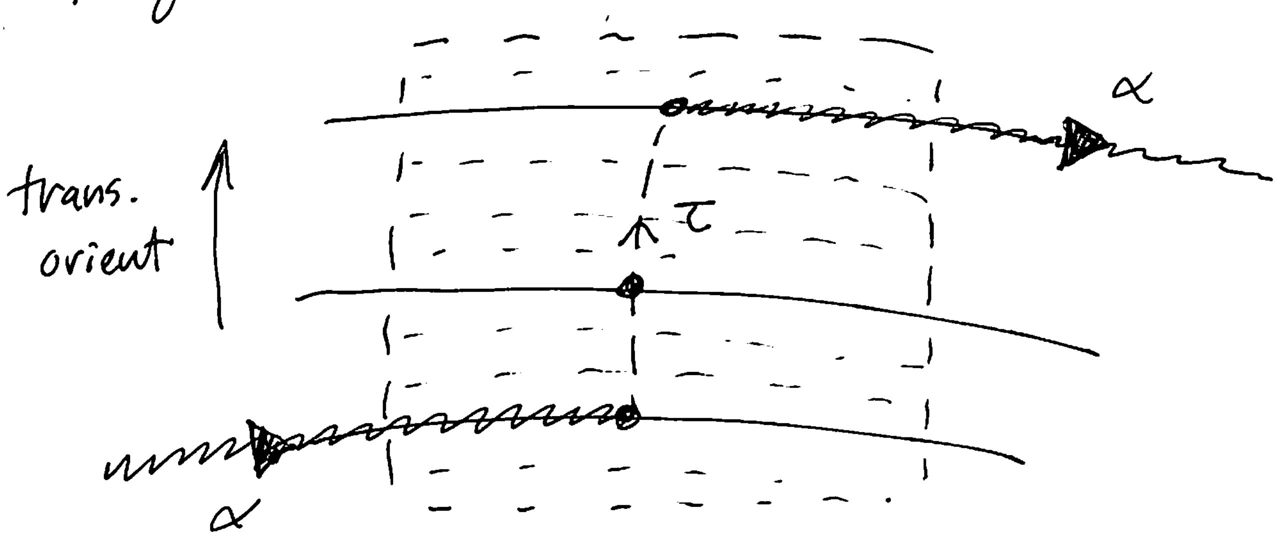


Clearly the plaque containing x_∞ is part of C.

To show the whole leaf containg x_∞ is in C, use holonomy. ▨

Lemma 2: Suppose $L_1 \subseteq \mathcal{F}$ is not closed. Then there is an embedded loop $\gamma \subseteq M$, trans. to $\mathcal{F}$ that meets L_1.

Pf: Will assume $\mathcal{F}$ is co-orient. Let U be a product chart that L meets in at least 3 plaques. Pick α in L joining the pts shown:



Now take γ to be a perturbation of $\alpha \cup \tau$. ▨

Lemma 3: [6.1.1 of Fol I] The union of cpt leaves of $\mathcal{F} \subseteq$ cpt M^3 is closed.

Pf idea: Suppose L_i are cpt, conv. to L. Set $V \subseteq H_2(M; \mathbb{Q}) = \operatorname{span}([L_i])$. As $\dim V < \infty$, can assume $[L_1], \dots, [L_m]$ span V.

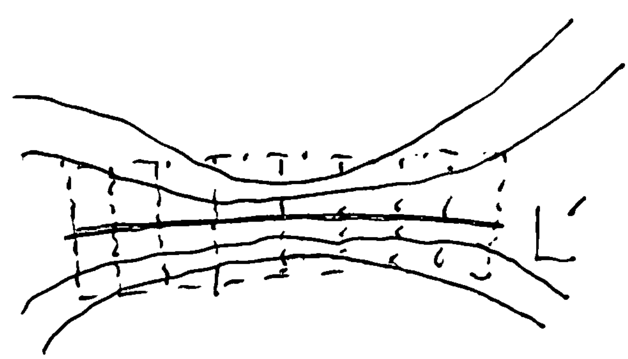
Take γ as in Lemma 2 disjoint from $\bigcup_{i=1}^{m} L_m$ and assume γ oriented by co-orient of $\mathcal{F}$.

There exists L_n with $L_n \cap \gamma \neq \emptyset$. In particular, $[L_n] \cap [\gamma] \neq 0$ but $[L_i] \cap [\gamma] = 0$ for $i \leq m$
$\Rightarrow [L_n] \notin \text{span}([L_1], \ldots, [L_m])$ a contradiction. $\blacksquare$

<u>Proof of Thm</u>: Since $\pi_1 S^2 = 1$, any $L \cong S^2$ has trivial holonomy and hence has a nbhd N where $\mathcal{F}$ is the prod. fol on $S^2 \times (-1,1)$. Let L' be a leaf in $\bar{N}$, say $L_i \in N \to L'$. By Lemma 3, L' is compact. Now L' has an (unfoliated) nbhd $N' \cong L' \times (-1,1)$ [trivial I-bundle as $\mathcal{F}$ co-orient] each $\{\ell'\} \times (-1,1)$ is a transv. for $\mathcal{F}$.

For $L_i \subseteq N'$ [i.e. all large i], proj. along

these transversals gives a covering map $L_i \to L'$.
As each $L_i \cong S^2$, have L' is S^2 or $\mathbb{RP}^2$.



Can't be $\mathbb{RP}^2$ as $\mathbb{RP}^2 \times (-1,1)$ is not orient.

So $L' \cong S^2$. Hence $\overline{N}$ is a union of S^2 leaves.

Hence the subset of S^2 leaves is both open and closed and so all of M. So $\mathcal{F}$ is a fol by S^2's.

Thus have $M = S^2 \times I \big/ {(v,1) \longrightarrow (f(v),0)}$

for some orientation reversing diff $f : S^2 \to S^2$.

Any such f is isotopic [smoothly homotopic through diffeo's] to id_{S^2} $\Rightarrow$ $M = S^2 \times S^1$ with the prod. fol. ▨

If time remains, discuss spinning fol. that are transverse to the boundary