Math 500: HW 3 due Friday, September 8, 2023.

Webpage: http://dunfield.info/500
Office hours: Wednesdays 1:30–2:30pm and Thursdays 2:00–3:00pm; additional times possible by appointment.
Textbook: Dummit and Foote, *Abstract Algebra*, 3rd edition.
Suplemental text: Charles Rezk, Lecture Notes for Math 500, posted on our course webpage.

1. Suppose $p, m, a \in \mathbb{Z}_{>0}$ with p prime. In this problem, you will show $\binom{p^a m}{p^a} \equiv m \bmod p$ using the Orbit Stabilizer Theorem. Consider the left action of $G = C_{p^a} = \langle r \rangle$ on $X = \{1, 2, \ldots, p^a m\}$ defined by setting $r \cdot k$ to be $k + m \pmod{p^a m}$. (If you view the points of X as arranged evenly along a circle, then r is rotation by $2\pi/p^a$.) Now let Y be the collection of all unordered subsets of X of size p^a, so that $|Y| = \binom{p^a m}{p^a}$. Use the induced action of G on Y to prove $|Y| \equiv m \bmod p$.

2. A subgroup $H \leq S_n$ of a symmetric group is said to be *transitive* if the evident action by H on $X = \{1, \ldots, n\}$ is a transitive action. Determine all transitive subgroups of S_4. You do not need to give a detailed proof that the list is complete.

3. Let G be a finite group of order $n = |G|$, and suppose there exists a *minimal non-trivial subgroup*, i.e., a subgroup $\{e\} \neq M \leq G$ such that $M \leq H$ for any $H \leq G$ with $H \neq \{e\}$. Show that G is not isomorphic to any subgroup of S_{n-1}. (Hint: Orbit Stabilizer Theorem.) Use this to show that the quaternion group of order 8 is not isomorphic to a subgroup of S_7.

4. Find all conjugacy classes and their sizes in the groups: D_8, Q_8, A_4.

5. Find all finite groups which have exactly two conjugacy classes. (The conjugacy class of the identity element counts as one of these.)

6. If G is a finite group of odd order, show that for $x \in G \setminus \{e\}$, the elements x and x^{-1} are not conjugate.

7. Consider $G' = GL_2(\mathbb{F}_3)$, and consider the elements

$$A := \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad B := \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, \quad C := \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}.$$

   Let $X = \{\{\pm A\}, \{\pm B\}, \{\pm C\}\}$, a set of subsets of G'. Show that conjugation induces an action by G' on X, and that this action is described by a surjective homomorphism $\phi\colon G' \to \mathrm{Sym}(X)$. Finally, identify $\mathrm{Ker}(\phi)$. (Hint: using Problem 1 on HW 2, it should be easy to understand what happens when G' is replaced by the subgroup $G = SL_2(\mathbb{F}_3)$.)

8. Show that $|\mathrm{Aut}(Q_8)| \leq 24$, by constraining the possible images of i, j under an automorphism. Then identify $\mathrm{Aut}(Q_8)$ using that Q_8 is isomorphic to a normal subgroup of $GL_2(\mathbb{F}_3)$. Determine $\mathrm{Inn}(Q_8)$ and $\mathrm{Out}(Q_8)$.

Credit: Problems 4, 5, 6, and 8 are from [DF] and Problems 2, 3, and 7 from [R].