Math 500: HW 7 due Friday, October 13, 2023.

Webpage: http://dunfield.info/500
Office hours: Wednesdays 1:30–2:30pm and Thursdays 2:00–3:00pm; additional times possible by appointment.

1. Let $R = \mathbb{Z}[\sqrt{2}] = \left\{ a + b\sqrt{2} \;\middle|\; a, b \in \mathbb{Z} \right\}$ as a subring of $\mathbb{R}$. Show that $x \in R^\times$ if and only if $N(x) \in \mathbb{Z}^\times$, where $N(a + b\sqrt{2}) = a^2 - 2b^2$. Then show that $R^\times$ has an element of infinite order. Hint: use that N is multiplicative.

2. Let $S = \mathbb{Z}[\sqrt{-3}] = \{a + b\sqrt{-3} \mid a, b \in \mathbb{Z}\}$. Determine the group of units $S^\times$. Then show that S is not a UFD, by finding an element with two inequivalent factorizations. Hint: use the function $N(a + b\sqrt{-3}) := \left|a + b\sqrt{-3}\right|^2 = a^2 + 3b^2$.

3. Assume R is a commutative ring with 1, and that for each $a \in R$ there is an $n > 1$ such that $a^n = a$. Show that every prime ideal of R is a maximal ideal.

4. Let A be a non-zero finitely generated ideal of a ring R. Use Zorn's Lemma to prove that there is an ideal B which is maximal with respect to the property that it does not contain A.

5. Prove that the ring $\mathcal{O}$ of quadratic integers in $\mathbb{Q}(\sqrt{2})$ is a Euclidean domain, using the function $a + b\sqrt{2} \mapsto \left|a^2 - 2b^2\right|$.

6. Prove that the quotient ring $\mathbb{Z}[i]/I$ is finite for every non-trivial ideal I. Hint: Use the fact that $I = (\alpha)$ for some nonzero α and the use the division algorithm in this Euclidean domain to see that every coset of I is represented by an element of norm less than $N(\alpha)$.

7. Prove that a quotient of a PID by a prime ideal is a PID.

8. Let $R = \mathbb{Z}[\sqrt{-n}]$, where n is a squarefree integer > 3.
   (a) Prove that 2, $\sqrt{-n}$, and $1 + \sqrt{-n}$ are irreducibles in R.
   (b) Prove that R is not a UFD. Hint: show that either $\sqrt{-n}$ or $1 + \sqrt{-n}$ is not prime.

9. (a) Prove that the quotient ring $\mathbb{Z}[i]/(1 + i)$ is a field of order 2.
   (b) Let $q \in \mathbb{Z}$ be a prime with $q \equiv 3 \bmod 4$. Prove that the quotient ring $\mathbb{Z}[i]/(q)$ is a field with q^2 elements.
   (c) Let $p \in \mathbb{Z}$ be a prime with $p \equiv 1 \bmod 4$ and write $p = \pi\overline{\pi}$ where π and its complex conjugate $\overline{\pi}$ are prime elements. Apply the Chinese Remainder Theorem (Section 7.6 of [DF] or Section 24 of [R2]) to see that $\mathbb{Z}[i]/(p) \cong \mathbb{Z}[i]/(\pi) \times \mathbb{Z}[i]/(\overline{\pi})$ as rings. Then show $\mathbb{Z}[i]/(p)$ has order p^2 and use this to conclude that $\mathbb{Z}[i]/\pi$ and $\mathbb{Z}[i]/\overline{\pi}$ are both fields or order p.

10. Determine all the ideals of the ring $\mathbb{Z}[x]/(2, x^3 + 1)$.

Credit: Problems 1–2 are from [R] and the rest from [DF].