Math 500: HW 10 due Friday, November 10, 2023.

Webpage: http://dunfield.info/500
Office hours: Wednesdays 1:30–2:30pm and Thursdays 2:00–3:00pm; additional times possible by appointment.

1. Let L/K be a finite field extension, and f an irreducible polynomial over K. Show that if $\deg f$ does not divide $[L : K]$, then f has no roots in L.

2. Show that $p = x^3 + 9x + 6$ is irreducible in $\mathbb{Q}[x]$. Let θ be a root of p, and find the inverse of $1 + \theta$ in $\mathbb{Q}(\theta)$.

3. Determine the minimal polynomial of $1 + i$ over $\mathbb{Q}$.

4. Let F be a subfield of $\mathbb{C}$. Show that for $D, E \in F$, we have $F(\sqrt{D}, \sqrt{E}) = F(\sqrt{D} + \sqrt{E})$. Hint: write $\alpha = \sqrt{D} + \sqrt{E}$, and express $\sqrt{D}$ and $\sqrt{E}$ as polynomials in α.

5. Let F be a subfield of $\mathbb{C}$. Show that for $D, E \in F$, we have $[F(\sqrt{D}, \sqrt{E}) : F] = 4$ if and only if $\sqrt{D}, \sqrt{E}, \sqrt{DE} \notin F$. Hint: if $\sqrt{E} \in F(\sqrt{D})$, then $\sqrt{E} = u + v\sqrt{D}$ for some $u, v \in F$. Use this to derive a contradiction.

6. Show that if $D = \pm p_1 \cdots p_r$ where $p_i \in \mathbb{Z}$ are distinct primes, then $\sqrt{D} \notin \mathbb{Q}$.

7. Show that if $p_1, \ldots, p_r$ are distinct primes, then $[\mathbb{Q}(\sqrt{p_1}, \ldots, \sqrt{p_r}) : \mathbb{Q}] = 2^r$. Hint: use induction on r and Problems 5 and 6 above.

8. Let F be a field of characteristic $\neq 2$. Let $a, b \in F$ with b not a square in F. Prove there exist $m, n \in F$ such that $\sqrt{a + \sqrt{b}} = \sqrt{m} + \sqrt{n}$ (in some extension field of F) if and only if $a^2 - b$ is a square in F. Hint: find a polynomial over F whose roots are $\{\sqrt{m} + \sqrt{n}, -\sqrt{m} + \sqrt{n}, \sqrt{m} - \sqrt{n}, -\sqrt{m} - \sqrt{n}\}$.

9. Determine the minimal polynomial of $\alpha = \sqrt{1 + \sqrt{2}}$ over $\mathbb{Q}$. In particular, explain why it has degree 4.

10. Prove that if $[F(\alpha) : F]$ is odd then $F(\alpha) = F(\alpha^2)$.

11. Let K/F be an algebraic extension and let R be a subring of K containing F. Show that R is a subfield of K containing F.

Credit: Problems 1, 6, 7, and 9 by Charles Rezk, rest from Dummit and Foote.