Lecture 10 Torsion in abelian groups; direct products

§38-41 of [R1], §5.1 of [DF].

Last time: G has the ascending chain condition (acc) when for every chain $H_1 \subseteq H_2 \subseteq H_3 \subseteq \cdots$ of subgps of G, there exists m such that $H_i = H_m$ for all $i \geq m$.

Thm: Every subgp of G is finitely generated (f.g.) $\Longleftrightarrow$ G has the acc.

Thm: $N \trianglelefteq G$. Then G has the acc $\Longleftrightarrow$ both N and G/N have the acc.

Thm: Any f.g. abelian G has the acc.

Cor: Any subgp of a f.g. abelian gp is also f.g.

Pf: We induct on the size n of the gen set for G.

$n=0$: $G = \{e\}$, done.

$n=1$: G cyclic. All subgps are cyclic, hence f.g. $\Rightarrow G$ has the acc.

General case: Suppose $G = \langle a_1, a_2, \dots, a_n \rangle$.

Set $N = \langle a_1, \dots, a_{n-1} \rangle$. As G abelian, $N \trianglelefteq G$.

By induction, N has the acc, as does $G/N = \langle a_n N \rangle$

Thus G also has the acc. $\blacksquare$

A $g \in G$ is <u>torsion</u> when $|g| < \infty$. The set of such elts is G_{tors}.

<u>Ex</u>: $\overbrace{G_{tors} = G}^{\text{torsion group}}$ when G is finite.

<u>Ex</u>: $\underbrace{G_{tors} = \{e\}}_{\text{torsion free group}}$ when G is free (by HW 2).

<u>Ex</u>: For $G = SL_2\mathbb{Z}$, G_{tors} is not a subgp! on HW1 had $A, B \in G$ with $|A| = 4$, $|B| = 3$ but $|AB| = \infty$.

<u>Thm</u>: G abelian $\Rightarrow G_{tors} \leq G$.

<u>Pf</u>: Key pt is $(g_1 g_2)^n = g_1^n g_2^n$, you can work out the details. $\square$

<u>Ex</u>: $G = \mathbb{Q}/\mathbb{Z}$ is all torsion since $x = \frac{a}{b} + \mathbb{Z} \in G$ has order b (assuming $\gcd(a,b) = 1$, $b > 0$).

<u>Thm</u>: Any f.g. abelian torsion gp is finite.

<u>Pf</u>: Suppose $G = \langle a_1, \ldots, a_\ell \rangle$. Then any $g \in G$ is $a_1^{e_1} \cdots a_\ell^{e_\ell}$ where $1 \leq e_i \leq |a_i|$ so $|G| \leq \prod |a_i|$. $\blacksquare$

Cor: $\mathbb{Q}$ and $\mathbb{Q}/\mathbb{Z}$ are not f.g.

Pf: $\mathbb{Q}/\mathbb{Z}$ torsion but not finite, so not f.g. by thm.
$\mathbb{Q}$ can't be f.g. as it has a non-f.g. quotient. $\square$

Thm: If G is abelian, G/G_{tors} is torsion-free.

Pf: Suppose xG_{tors} had order $n < \infty$. Then $x^n \in G_{tors}$
$\Rightarrow x^n = y$ where $|y| = m < \infty$. Then $x^{nm} = 1 \Rightarrow x \in G_{tors}$. $\square$

[Before getting to class. of all f.g. abelian gps, need to talk about products.]

Suppose $G_1, \ldots, G_n$ are groups [poss. non abelian]
Their (direct) product is $G := G_1 \times \cdots \times G_n$
$= \{(g_1, \ldots, g_n) \mid g_i \in G_i\}$ with group operation.

$$(x_1, \ldots, x_n) \cdot (y_1, \ldots, y_n) = (x_1 y_1, \ldots, x_n y_n).$$

Projection homom. $\pi_k : G \twoheadrightarrow G_k$
$$(x_1, \ldots, x_n) \longmapsto x_k$$

Inclusion hom: $i_k : G_k \hookrightarrow G$
$$g_k \longmapsto (e, \ldots, e, g_k, e, \ldots, e)$$
k^{th} place.

The coordinate axis $G_k' = i_k(G_k)$ is isom to G_k.

Note: For $k \neq l$, anything in G_k' commutes with anything in G_l'. Thus $G_k' \trianglelefteq G$ as $G = \langle \cup G_i \rangle$.

[Ex: $F(a,b) \times F(c,d)$ has an interesting mix of (non) commuting behavior.]

Thm: Suppose $G_1, G_2, \dots, G_n \trianglelefteq G$ with

1) $G_1 G_2 \cdots G_n = G$.
2) $G_k \cap (G_1 G_2 \cdots G_{k-1}) = \{e\}$

Then $G_1 \times G_2 \times \cdots \times G_n \cong G$ by

$$(g_1, g_2, \dots, g_n) \longmapsto g_1 g_2 \cdots g_n.$$

Pf: See §41 of [RI] or §5.1 of [DF].

Thm: If $g \in G_1 \times G_2 \times \cdots \times G_n$, then $|g| = \operatorname{lcm}\{|g_i|\}$ where the latter is ∞ if any $|g_i|$ is.

Ex: $C_3 \times C_4 = \langle a \mid a^3 \rangle \times \langle b \mid b^4 \rangle$

is cyclic since (a,b) has order $\operatorname{lcm}(3,4) = 12$.