Lecture 17: Kinds of ideals; field of fractions

[R2]: §16–21
[DF]: 7.4–7.5

Last time: R comm ring with 1.

An ideal $M \subsetneq R$ is maximal when $M \subseteq I \subseteq R$ (ideal)
$\Rightarrow I = M$ or $I = R$.

Note: #7 on HW6 has been removed.

Thm: M is maximal $\Leftrightarrow R/M$ is a field.

Ex: $(x) \subseteq \mathbb{Q}[x]$; $(p) \subseteq \mathbb{Z}$

An ideal $P \subsetneq R$ is prime if whenever $a \cdot b \in P$ then $a \in P$ or $b \in P$ (or both).

Thm: R comm ring with 1. An ideal $P \subseteq R$ is prime $\Leftrightarrow R/P$ is an integral domain.

Cor: Maximal ideals are prime.

Pf: R/P is an int domain means whenever $x, y \in R/P$ have $x \cdot y = 0$ then $x = 0$ or $y = 0$.
If $x = a + P$ and $y = b + P$ then $x = 0 \Leftrightarrow a \in P$, $y = 0 \Leftrightarrow b \in P$, and $xy = 0 \Leftrightarrow ab \in P$. So the result follows. $\blacksquare$

Ex: $(x) \subseteq \mathbb{Z}[x]$ is prime but not maximal, as quot. is $\mathbb{Z}$. A max ideal containing (x) is $(2, x)$, since $\mathbb{Z}[x]/(2,x) \cong \mathbb{Z}/2\mathbb{Z}$ a field.

Thm: R comm ring with 1. Every proper ideal is contained in a maximal one.

Idea: Given $I_0 \subsetneq R$. If I_0 is not maximal, choose an ideal $I_0 \subsetneq I_1 \subsetneq R$. Repeating, we either find the needed max ideal or build a chain

$$I_0 \subsetneq I_1 \subsetneq I_2 \subsetneq \cdots \qquad \subsetneq R.$$

By HW, $J_0 = \bigcup_{k=1}^{\infty} I_k$ is an ideal. It is not R as $J_0 = R \Leftrightarrow 1 \in J_0 \Leftrightarrow 1 \in I_k$ for some k $\Leftrightarrow$ some $I_k = R$. If J_0 is maximal, great, otherwise find $J_0 \subsetneq J_1 \subsetneq R \ldots$

Need to use transfinite induction + axiom of choice in the form of:

> Zorn's Lemma: Suppose X is a partially ordered set where every nonempty chain has an upper bound. Then X has a maximal elt.

Apply to $X = \{ J \text{ ideal with } I_0 \subsetneq J \subsetneq R \}$.

See §17-18 of [R2] for details.

Motivation $\mathbb{Z} \rightsquigarrow \mathbb{Q} = \{\text{fractions } a/b \text{ for } a,b \in \mathbb{Z}\}$

R int. domain, set

$$\mathrm{Frac}(R) = \{(r,d) \in R \times R \mid d \neq 0\} / \sim$$

where $(r,d) \sim (r',d') \Leftrightarrow rd' = r'd$. Ops are

$$(r,d) + (r',d') = (rd' + dr', dd')$$
$$(r,d) \cdot (r',d') = (rr', dd').$$

Notes: ① $R \hookrightarrow \mathrm{Frac}(R)$ is an inj. ring hom.
$r \longmapsto (r,1)$

See text for why this all makes sense.

② $0 = (0,1)$ and $1 = (1,1)$.

③ $\mathrm{Frac}(R)$ is a field: $(r,d)^{-1} = (d,r)$ for $r \neq 0$.
field of fractions.

④ Write elts as r/d or $[r/d]$.

Ex: $\mathbb{Q} = \mathrm{Frac}(\mathbb{Z})$.

Ex: $\mathrm{Frac}(\mathbb{Q}[x]) =$ field of rat'l fns. Typical elts are

$$\frac{3x^2 - 2x + 1}{\frac{1}{3}x^3 + 5x + 6/5}, \quad \frac{x^3 - 1}{x^2 - 3x + 2} = \frac{x^2 + x + 1}{x - 2}$$

(mult top/bot. by $(x-1)$)

R comm ring with 1.

$D \subseteq R$ is <u>multiplicatively closed</u> if $1 \in D$ and $a, b \in D \Rightarrow ab \in D$.

<u>Ex 1</u>: $D = \{2^n \mid n \in \mathbb{Z}_{\geq 0}\} \subseteq \mathbb{Z}$

<u>Ex 2</u>: $D = \{x^n \mid n \in \mathbb{Z}_{\geq 0}\} \subseteq \mathbb{Q}[x]$

<u>Ex 3</u>: $D = R \setminus \{0\}$, provided R is an int. domain.

The <u>ring of fractions</u> is

$$D^{-1}R := R \times D / \sim \qquad (r, d) \sim (r', d') \iff$$
$$m \in D \text{ with } m(rd' - r'd) = 0.$$

<u>Ex 1</u>: $D^{-1}R = \left\{ \frac{a}{2^n} \mid a \in \mathbb{Z}, n \geq 0 \right\} \subseteq \mathbb{Q}$.

<u>Ex</u>: 2 $D^{-1}R =$ Laurent polys such as $3x^{-5} + \frac{1}{2}x^{-2} + x + 4x^2$

$= \left\{ \sum_{k \in \mathbb{Z}} a_k x^k \mid a_k \in R, \text{ all but finitely many are } 0 \right\}$

<u>Ex 3</u> $\mathrm{Frac}(R)$.

<u>Ex 4</u> $P \subseteq R$ prime, set $D = R \setminus P$. Then R_P is the <u>localization</u> of R at P.

<u>Note</u>: When D contains 0-divisors, the map $R \to D^{-1}R$, $r \mapsto [r/1]$ can have kernel.

See §19-21 of [R2].