Lecture 25: Free modules, torsion, linear (in)dependence.

§52, 56-57 of [R2]
§10.3 of [DF]

Last time: R ring with 1. An R-module M is free on a subset $S \subseteq M$ when $\forall m \in M$ $\exists! \{a_s \in R\}_{s \in S}$ where $a_s \neq 0$ for only finitely many s and $m = \sum_{s \in S} a_s \cdot s$.

[If R is a field, we call such S a basis]

Ex: $R^n := \bigoplus_{i=1}^{n} R$ is free on $\{e_1, \dots, e_n\}$ with $e_i = (0, \dots, 0, 1_R, 0, \dots, 0)$ ← ith place.

Given any set S and ring R can construct a free R-module on S: [Compare with free gps.]

$$M = \Big\{ \underbrace{\sum_{s \in S} a_s s}_{\text{formal sum}} \;\Big|\; a_s \in R, \text{ only finitely many} \neq 0 \Big\} \cong \bigoplus_{s \in S} R$$

Universal Prop: Suppose an R-module M is free on $S \subseteq M$. Given a fn $\phi: S \longrightarrow N$, with N an R-mod, $\exists!$ hom of R-mods $\tilde{\phi}: S \to N$ with $\tilde{\phi}(s) = \phi(s)$ for all $s \in S$.

Pf idea: Given $m \in M$, write $m = \sum_{s \in S} a_s \cdot s$ for $a_s \in R$ unique per the def. Define $\tilde{\phi}(m) := \sum_{s \in S} a_s \, \phi(s)$ and check that this works. ▨

Ex: $R = \mathbb{Z}$. $M = \mathbb{Z} \oplus \mathbb{Z}$ is free on $S = \{(1,0), (0,1)\}$ but not free on $\{(1,1)\}$ or $\{(1,0),(0,1),(1,1)\}$

If $(M,+)$ has an elt m of finite order, e.g. $\mathbb{Z}/6 \oplus \mathbb{Z}$, then it is not free on any subset. (HW!)

Ex: R = field $\Rightarrow$ all R-modules are free.

R an int domain, M an R-module. An $m \in M$ is torsion when $\exists r \neq 0$ in R with $r \cdot m = 0$.

Ex: 0_M always torsion (take $r = 1_R$).

Ex: $R = \mathbb{Z}$. Then $m \in M$ is torsion $\Longleftrightarrow$ $|m| < \infty$ in $(M,+)$.

Lemma: $M_{tors} := \{m \in M \mid m \text{ torsion}\} \subseteq M$ is a submodule.

Pf: Suppose $m, n \in M_{tors}$. Pick r, s nonzero with $rm = 0 = sn$. To show $am + bn \in M_{tors}$ for all $a, b \in R$. Consider $(rs)(am+bn) =$
$(rsa)m + (rsb)n = (as)(r \cdot m) + (br) \cdot (s \cdot n) = 0 + 0 = 0$.
As R is an int domain, $rs \neq 0 \Rightarrow am + bn \in M_{tors}$. ▨

Say M is torsion when $M_{tors} = M$ and torsion-free when $M_{tors} = \{0\}$.

Ex: $R = \mathbb{Z}$; $M_1 = \mathbb{Z}/3 \oplus \mathbb{Z}/6$ is torsion; $M_2 = \mathbb{Z}^5$ is torsion free
$M_3 = \mathbb{Z}/2 \oplus \mathbb{Z}$ is neither.

Ex: Any free module is torsion-free. (HW!).

Ex: A cyclic module M is torsion $\Longleftrightarrow M \not\cong R$.

Pf: Recall $M \cong R/I$ for some ideal I. If $I \neq \{0\}$ then pick $i \in I$ nonzero and note $i \cdot (m + I) =$
$im + I = 0 + I$. for any $m \in R$.

Ex: M/M_{tors} is torsion-free.

Pf: (Skip!) Suppose $r \cdot (m + M_{tors}) = 0$ for $r \neq 0$ in R

Then $r \cdot m \in M_{tors} \Rightarrow \exists s \neq 0$ in R with $s \cdot (r \cdot m) = 0 \Rightarrow (sr) \cdot m = 0 \Rightarrow m \in M_{tors} \Rightarrow$ $m + M_{tors} = 0$. $\blacksquare$

R int domain, M an R-mod. A subset $S \subseteq M$ is <u>R-linearly dependent</u> where $\exists\, K \geq 1$ and $s_1, \ldots, s_K \in S$ and $r_1, \ldots, r_K \in R$ nonzero with $\sum_{i=1}^{K} r_i s_i = 0$. Otherwise, S is <u>R-linearly independent</u>.

<u>Ex</u>: If M is free on S then S is R-lin indep.

<u>Ex</u>: If M is torsion, only R-lin indep subset is $\emptyset$.

For any M, consider $\mathcal{C} = \{ S \subseteq M \mid S \text{ } R\text{-lin indep.} \}$ ordered by inclusion.

<u>Thm</u>: M module over int. domain R. Then M has a maximal R-lin indep set.

<u>Pf</u>: Zorn's Lemma. $\blacksquare$

<u>Lemma:</u> An R-lin indep $S \subseteq M$ is maximal $\iff M/_{RS}$ is a torsion module.

<u>Pf</u> (Skip!) Set $N = RS$. Let $y \in M$ with $\bar{y} = y + N \in M/_N$. Now $\bar{y}$ is torsion $\iff \exists b \neq 0$ in R with $by = a_1 s_1 + \cdots + a_n s_n$ where $a_i \neq 0$ in R and s_i are distinct elts of S.

$\iff y \in S$ or $S \cup \{y\}$ is R-lin dep.

So all $\bar{y}$ are torsion $\iff S$ is a maximal R-linear indep set. $\blacksquare$

<u>Cor:</u> Any vector space M over a field R has a basis.

<u>Pf:</u> Let $S \subseteq M$ be a max R-lin indep set.

Then $M/_{RS}$ is torsion $\underset{R \text{ field}}{\Longrightarrow} M/_{RS} = \{0\}$

$\Rightarrow M = RS$ and S is a basis. $\blacksquare$