Lecture 4: Group presentations and group actions

Previously: S set

$F(S) :=$ free group of reduced words on $S \cup \{s^* \mid s \in S\}$

Ex: $F(a,b)$ $\quad (aba^{-1}ba)\cdot(a^{-1}b^{-1}a^2b^2) = abab^2$

Theorem: Consider $F(s_1, s_2, \ldots, s_n)$. For all groups G and elts $g_1, \ldots, g_n \in G$, $\exists!$ hom. $\phi : F(s_1, \ldots, s_n) \to G$ with $\phi(s_i) = g_i$.

Contrast: ① $\nexists$ $C_3 = \langle a \rangle \xrightarrow[\text{hom}]{\phi} C_5 = \langle b \rangle$ with $\phi(a) = b$

since $e = \phi(e) = \phi(a^3) = (\phi(a))^3 = b^3 \neq e$.

② $\nexists$ $S_3 \xrightarrow{\phi} C_6 = \langle x \rangle$ with $\phi((12)) = x^3$
$\phi((123)) = x^2$.

A group presentation is a pair (S, R) where $R \subseteq F(S)$. The group presented by this data is

$$\langle S \mid R \rangle := F(S) / \langle\langle R \rangle\rangle$$

where $\langle\langle R \rangle\rangle :=$ normal closure of R $= \left\langle \bigcup_{g \in F(S)} gRg^{-1} \right\rangle$ = smallest normal subgp contain. R.

Ex: $\langle a \mid a^n \rangle = \underbrace{F(a)}_{C_\infty} / \underbrace{\langle a^n \rangle}_{R} \cong C_n$.

Ex: $G = \langle a, b \mid \overbrace{a^2, b^2, (ab)^2}^{R} \rangle$

Here $\langle\langle R \rangle\rangle = \{ ba^2b^{-1}, a^2b^2, (ab)^2b^{-2} = abab^{-1}, \dots \}$

Claim: $G \cong \underset{\langle x \rangle}{\underset{\parallel}{C_2}} \times \underset{\langle y \rangle}{\underset{\parallel}{C_2}} = \{ (e,e), (x,e), (e,y), (x,y) \}$

Pf: Let $\phi : F(a,b) \longrightarrow C_2 \times C_2$ be the hom. with $\phi(a) = (x,e)$ and $\phi(b) = (e,y)$. Now

$\phi(a^2) = \phi(a)\,\phi(a) = (x,e)\cdot(x,e) = (x^2, e) = (e,e)$

and similarly for $\phi(b^2)$ and $\phi((ab)^2)$. Hence $\langle\langle R \rangle\rangle \subseteq \operatorname{Ker} \phi$. By the homom. lemma,

$\exists$ a hom $\bar{\phi} : G \longrightarrow C_2 \times C_2$, which is surjective. To see its an isomorphism, we show $G = \{e, a, b, ab\}$. In G, $a^2 = 1$ $\Rightarrow a^{-1} = a$ and sim $b^{-1} = b$. As $abab = 1 \Rightarrow$ $ab = ba$ so G is abelian. A given $g \in G$ as a word in $F(a,b)$ can now be rewritten, e.g.

$$g = aba^{-3}b^2aba^2 = a^{1-3+1+2}b^{1+2+1} = a$$

to be on of $\{e, a, b, ab\}$. $\blacksquare$

Some more examples (no proofs!)

Ex: $D_{2n} \cong \langle r, s \mid r^n, s^2, srsr \rangle$

Ex: $SL_2\mathbb{Z} \cong \langle a, b \mid a^4, b^6, a^2b^{-3} \rangle$

Ex:

$$S_n \cong \left\langle s_1, \dots, s_{n-1} \;\middle|\; \begin{array}{ll} s_i^2 & \text{for all } i \\ (s_is_j)^2 & \text{for all } |i-j| \geq 2. \\ (s_is_j)^3 & \text{for all } |i-j| = 1 \end{array} \right\rangle$$

Idea $s_i \longmapsto (i \;\; i{+}1)$

Note: Presentations are tricky in general, e.g. no algorithm to compute $|\langle x_1, \dots, x_n \mid r_1, \dots, r_m \rangle|$.

Suppose G is a group, X a set.

A (<u>left</u>) <u>group action</u> is a map $G \times X \longrightarrow X$, $(g, x) \longmapsto g \cdot x$

where

① $g_1 \cdot (g_2 \cdot x) = (g_1 g_2) \cdot x \quad \forall g_1, g_2 \in G, x \in X.$

② $e \cdot x = x \quad \forall x \in X.$

<u>Ex</u>: S_n acts on $\{1, 2, \ldots, n\}$ by $\sigma \cdot i = \sigma(i)$

<u>Ex</u>: $GL_n \mathbb{F}$ acts on $\mathbb{F}^n$ by $A \cdot v =$ matrix mult of A on col vect v.

<u>Ex</u>: For $H \leq G$, the gp G acts on G/H by

$$g \cdot (xH) = (gx)H.$$

<u>Ex</u>: A group G acts on itself by conjugation

$$g \cdot x = g \cdot x \cdot g^{-1}$$

<u>Ex</u>: D_{2n} acts on $X = \{\text{pts of reg. } n\text{-gon in } \mathbb{R}^2\}$

<u>Ex</u>: $SL_2 \mathbb{R}$ acts on $\mathbb{H} = \{z \in \mathbb{C} \mid \operatorname{Im}(z) > 0\}$

by $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z = \dfrac{az+b}{cz+d}$. (Really!)