Lecture 6: Applied Group actions

Sec 4.2–3 of [DF]
Sec 21–25 of [R]

Previously... $G \curvearrowright X$

Stabilizer: $G_x := \{ g \in G \mid g \cdot x = x \}$

Orbit: $G \cdot x := \{ g \cdot x \mid g \in G \}$

Thm: For $x \in X$, have a bijection $G/G_x \longrightarrow G \cdot x$.

Thus $|G \cdot x| = [G : G_x]$ and $|G| = |G \cdot x| |G_x|$.

When $G \curvearrowright X$, X is partitioned into disjoint orbits $\mathcal{O}_1, \mathcal{O}_2, \dots$ (If define $x \sim x'$ when $\exists g \in G$ with $gx = x'$, then the orbits are the equivalence classes of this relation.)

Cor: Suppose $G \curvearrowright X$ with $|X| < \infty$. Let $\mathcal{O}_1, \dots, \mathcal{O}_k$ be the orbits and pick $x_i \in \mathcal{O}_i$.
Then $|X| = \sum_{i=1}^{k} |\mathcal{O}_i| = \sum_{i=1}^{k} [G : G_{x_i}]$.

[Now some applications... building to Sylow thms.]

Cayley's Thm: Any group G is isomorphic to a subgp of some $\mathrm{Sym}(X)$.

Pf: $G \curvearrowright G$ by left-mult, giving a homomorphism $G \longrightarrow \mathrm{Sym}(G)$. This is injective as if $\phi_g = id_G$ have $g \cdot e = e \Rightarrow g = e$. ▨

Cor: When $|G| < \infty$, then $G \leq S_{|G|}$.

Conjugation action: $G \curvearrowright G$ by $g \circ h = ghg^{-1}$

Orbits are called conjugacy classes.

$$Cl(h) := \{ghg^{-1} \mid g \in G\}$$

The stabilizer of h is called its centralizer

$$C_G(h) = \{g \in G \mid ghg^{-1} = h \Leftrightarrow gh = hg\}$$

The kernel of $G \longrightarrow \mathrm{Sym}(G)$ is the center

$$Z_G = \bigcap_{h \in G} C_G(h) = \{g \in G \mid gh = hg \ \forall h \in G\}$$

Ex: $Cl(e) = \{e\}$ and $C_G(e) = G$.

Ex: When G is abelian $Cl(g) = \{g\}$.

Ex: $G = S_n$ $\quad Cl((12)) = \{$all transpositions $(12), (13), (23), \dots\}$

For general $\sigma \in S_n$, if we write it as a prod. of disjoint cycles, e.g. $\sigma = (12)(45)(637)$ then $Cl(\sigma)$ = all perm. with the same cycle decomposition.

Class equation thm: For a finite gp G,

$$|G| = |Z_G| + \sum_{k=1}^{r} [G : C_G(g_k)]$$

where $g_1, \ldots, g_k$ are rep's of the conj classes not contained in Z_G.

Pf: Two kinds of orbits under conj. action:

1) Those of size $1 \iff$ contained in Z_G
2) those of size $\geq 2 \iff$ not contained in Z_G.

Now apply the cor on page 1. ▨

p - prime. A p-group is one where $|G| = p^a$ with $a \geq 0$.

Thm: A p-group G has $Z_G \neq \{e\}$.

Pf: Have $|Z_G| = \underset{\underset{p^a}{\shortparallel}}{|G|} - \sum \underbrace{[G : C_G(g_k)]}_{\neq 1,\text{ divides } |G| \Rightarrow \text{divisible by } p}$

and so $p \mid |Z_G| \Rightarrow Z_G \neq \{e\}$. ▨

Thm: If $|G| = p$ then $G \cong C_p$.

Pf: Pick $g \neq e$ in G. Then $|g| > 1$ and $|g| \mid |G|$ gives $|g| = p$ and $\langle g \rangle = G$ as needed. □

Thm: If $|G| = p^2$ then G is abelian.

Lemma: G a group with G/Z_G cyclic. Then G is abelian.

Pf: Pick $g \in G$ whose projection generates G/Z_G. So each elt in G is $g^k z$ with $k \in \mathbb{Z}$ and $z \in Z_G$. Now $(g^i x)(g^j y) = g^{i+j} xy = (g^j x)(g^i y)$ for $x, y \in Z_G$, so G is abelian ▨

Pf of Thm: As $Z_G \neq \{e\}$, have $|Z_G| = p$ or p^2. If p^2 then G is abelian. If $|Z_G| = p$ then $|G/Z_G| = p \Rightarrow G/Z_G$ is cyclic $\Rightarrow G$ abelian. ▨